Math Club Talks

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The UT Math Club meets weekly and invites speakers to give talks every Tuesday at 5:00 PM! Here are some notes I've TEX'd up from some of them (not all). Source: https://git.simonxiang.xyz/math_notes/files.html

Contents

Lecture 1

All voting systems are fundamentally screwed (9/08/20)

Today's speaker is Tom Gannon, a Ph.D student who made his entire website without any help from outsite sources. We'll be talking about why all voting systems are fundamentally screwed! (AKA, Arrow's impossibility theorem)[1].

Note. I'm actually taking the notes on November 2nd, 2020. Although I did attend this talk, I didn't take notes because I was still figuring out my live-TEX setup. Now seems like an appropiate time to revisit this topic, given that my government test is today and the election is tomorrow.

1.1 What's a voting system?

Let's declare a list of *alternatives* to choose from. It's not always just voting for the thing that's on top, we make a total preference. Informally, a voting system take as input the following:

- A *personal preference* list of the alternatives for each person.

And output the following:

- A *societal preference list* of the alternatives.

Example 1.1. Who should be the math club president? Whoops, they are the president and vice-president.

- *Alternatives:* {Ryan, Shannon}
- *Voters:* The people in this Zoom call

So how are we gonna rank them? Here are some of our options for a voting system.

- **First Past the Post**: Order the societal preference list by how many times the person appeared as the top preference.
- **'Weighted FPTP'**: Same as FPTP, but the current president's vote counts for two votes.
- **Last Past the Post**: This isn't a real system, but serves to demonstrate how things can get wonky when we're allowed to change people's parameters: order the societal preference list by who got the least votes. By definition, this is a voting system.
- **Dictatorship**: Declare the societal preference list to be the same as the dictator's preference list.

1.2 The Pareto condition and independence of irrelevant alternatives

There are some properties of voting systems: the first thing we want is something called the Pareto condition.
Definition 1.1 (Pareto condition)**.** If, when every person puts in the *same* personal preference list, the voting system returns that list as the societal preference list, we say a voting system satisfies **the Pareto condition**.

- *Examples:* First Past the Post, 'Weighted FPTP', Dictatorships
- *Non-examples:* Last Past the Post

Example 1.2. Our new question: who should win the 2000 election?

- *Alternatives:* {George W. Bush, Al Gore, Ralph Nader}
- *Voters:* Eligible Florida residents

[1] Sorry for the clickbait title.

Candidate	Number of Votes
George W. Bush	2,912,790
Al Gore	2,912,253
Ralph Nader	97,488

Figure 1: FPTP Election Results, Florida 2000

This is an example of something of mathematical interest: if everyone who voted for Ralph Nader had Al Gore as their second choice, then if we took into account a ranked preference list the outcome of the election would have been totally different. This is called **Instant Runoff Voting—** here voters rank all preferences. Declare the person who got the least amount of votes last on the preference list, eliminate them, and repeat.

So the Pareto condition is a weak condition that's really easy to satisfy. Pretty much any reasonable voting system should satisfy it. Now let's introduce another reasonable condition that things should satisfy.

Definition 1.2 (Independence of irrelevant alternatives)**.** We say that a voting system is **independent of irrelevant alternatives** if (informally) for every pair of alternatives x, y we can know the relevant position of x and y on the societal preference list just from knowing the relative position of x and y on all the individual's preference list.

As an exercise, Tom suggest defining voting systems and independence of irrelevant alternatives using set theory. Let's try it:

Definition 1.3. Let A be an ordered set of alternatives. Then if $x \in X$ where X is the set of people, then a *personal preference list* is a permutation of A, denoted σ_x. Then a **voting system** is a function $f : S_X \to S_A$, where S_A denotes the permutation group on A, and S_X denotes the set of set of personal preferences for all $x \in X$. By this definition, a voting system satisfies the pareto condition if for all $x_1, x_2 \in X$, $\sigma_{x_1} = \sigma_{x_2}$, then $f(\sigma_{x_1} = \sigma_{x_2}) = \sigma_{x_1} = \sigma_{x_2} \in S_A$, and is **independent of irrelevant alternatives** if for all $x, y \in A$, the order of x and y in: wait, this entire definition is screwed, I'm only taking in one permutation and returning the societal preference list. Why helpppppp

OK, I looked it up, I was kind of close. The idea was to use the set of all total orders on a set of outcomes.

Definition 1.4 (Actual definition)**.** Let A be a set of outcomes and N a set of voters. Denote the set of all full linear orderings of A by $L(A)$. Then a (strict) **social welfare function** (preference aggregation rule or voting system) is a function

$$F : L(A)^N \to L(A)$$

that aggregates voter's preferences into a single preference order on A. An N-tuple $(R_1, \cdots, R_N) \in L(A)^N$ of voters preferences is called a *preference profile*. Here's how we precisely define the stuff we talked about earlier:

- **Pareto condition** (unanimity): if an alternative a is strictly greater than b for all total orderings $R_1, \cdots, R_N$, then a is strictly greater than b by $F(R_1, R_2, \cdots, R_N)$.
- **Dictatorships**: A *dictator* is an individual i such that for all $(R_1, \cdots, R_N) \in L(A)^N$, a strictly greater than b by R_i implies that a strictly greater than b by $F(R_1, R_2, \cdots, R_N)$ *for all* a, b.
- **Independence of irrelevant alternatives**: For two preference profiles $(R_1, \cdots, R_N)$ and $(S_1, \cdots, S_N)$ such that for all individuals i, alternatives a and b have the same order in R_i as in S_i, then alternatives a and b have the same order in $F(R_1, \cdots, R_N)$ as in $F(S_1, \cdots, S_N)$.

Question: Is there a voting system which satisfies both the pareto condition and is independent of irrelevant alternatives?
Answer: Yes, a dictatorship! Just look at the dictator's list.

1.3 Arrow's Impossibility Theorem

Well, are there any other voting systems that satisfy these two conditions then? The answer turns out to be no!

Theorem 1.1 (Arrow's impossibility theorem)**.** *Assume that V is a voting system with more than two alternatives which satisfies the Pareto condition and is independent of irrelevant alternatives. Then V is a dictatorship.*

As a corollary, the contrapositive of this theorem says that there are no voting systems with more than two alternatives which satisfy the Pareto condition, are independent of irrelevant alternatives, and are not a dictatorship.

Proposition 1.1 (No ties)**.** *Assume we have a voting ssytem with more than two alternatives which satisfies Pareto and IIA. Then the voting system can produce no ties.*

Proof. We do this by contradiction.

If $\begin{array}{c|c} \text{Left side of Room} & \text{Right side of Room} \\ \hline a > b & b > a \end{array} \mapsto a = b,$

then $\begin{array}{c|c} \text{Left side of Room} & \text{Right side of Room} \\ \hline a > c > b & c > b > a \end{array} \mapsto c > b = a,$

and $\begin{array}{c|c} \text{Left side of Room} & \text{Right side of Room} \\ \hline a > b > c & b > c > a \end{array} \mapsto a = b > c.$

By IIA, $\begin{array}{c|c} \text{Left side of Room} & \text{Right side of Room} \\ \hline a > c & c > a \end{array} \mapsto c > a$ and $a > c$, a contradiction.

Therefore, there are no ties. ※ ⊠

Definition 1.5 (Dictating sets)**.** We say a subset S of voters are a **dictating set** if, whenever everyone in S puts in the same personal preference list into the voting system, that list is the societal preference list, regardless of what anyone else votes.

- *Example:* The set of all voters is a dictating set by Pareto.
- *Note:* If S is a set with one element, then S is a dictating set if and only if the person in S is a dictator.

Definition 1.6 (Monotonic)**.** A voting system is **monotonic** if for all alternatives a, b, the following property holds: If

$$\begin{array}{c|c} \text{Left side of Room} & \text{Right side of Room} \\ \hline a > b & b > a \end{array} \mapsto a > b.$$

and some people in the 'everyone else' part switch their vote to $a > b$ then the societal preference list still has $a > b$.

Basically, this allows us to focus on the worst case scenario.

Definition 1.7 (Forcing)**.** Given two alternatives a, b, we say a subset S of voters can **force** $a > b$ if

$$\begin{array}{c|c} \text{People in S} & \text{Everyone Else} \\ \hline a > b & ??? \end{array} \mapsto a > b.$$

Difference between forcing and a dictating set? Forcing is for two specific alternatives, while dictating is about everything else.

1.4 Proof of Arrow's impossibility theorem

The proof will follow from a lemma and a proposition:

Lemma 1.1 (Forcing lemma)**.** *If a subset of voters X that can force $a > b$ and we partition $X = L \amalg M$, then for any third alternative c, either L can force $a > c$ or M can force $c > b$.*

Proposition 1.2. *If X can force some element a over some element b, then X can force* any *element over* any *other element, i.e. X is a dictating set.*

To see why these two things would prove the theorem, let X be the entire set for the forcing lemma by Pareto (pareto implies that X is a dictating set): then after a disjoint partition, either it becomes a dictating set or the others do. So there exists a strictly smaller dictating set, since X is finite repeat until we have a dictator.

Proof. Here's the proof of the forcing lemma, the core essence of the proof is using the fact that we have no ties. We know $\begin{array}{c|c|c} \text{L} & \text{M} & \text{Everyone Else} \\ \hline a > b > c & c > a > b & b > c > a \end{array} \mapsto a > b$ (but we don't know where c lies). We get that either $c > a > b$ and $a > c$ (by no ties): if $c > a > b$, then M can force $c > b$ by definition. If the output is $a > c$, then L forces $a > c$ by defintion.

Question: What happens if $M = \emptyset$? If $L = \emptyset$? ⊠

Corollary 1.1 (Forcing lemma for $\emptyset$)**.** *If a subset X can force $a > b$ for two distinct alternatives a, b, then X can force $a > c$ and $c > b$ for any third alternative $c \notin \{a, b\}$.*

Corollary 1.2. *If a subset X can force $a > b$ for two distinct alternatives a, b, then X can force $b > a$.*

Proof.

- X can force $a > b \underset{\text{Forcing } a>b}{\Longrightarrow} X$ can force $a > c$.
- X can force $a > c \underset{\text{Forcing } a>c}{\Longrightarrow} X$ can force $b > c$.
- X can force $b > c \underset{\text{Forcing } b>c}{\Longrightarrow} X$ can force $b > a$.

⊠

Problem. *Use the forcing lemma to show that if X can force $a > b$ then for* any *distinct alternatives c, d, X can force $c > d$.*

Proof. Let A be the set of alternatives, assume we can force $a > b$. Then by Corollary 1.1, we can force $a > c$ and $c > b$ for some alternative $c \notin \{a, b\}$. Apply Corollary 1.1 again to get that $a > d$ and $d > c$ for some alternative $d \notin \{a, b, c\}$, given $a > c$. But then by Corollary 1.2, we can "swap" d and c to get $c > d$, and this applies for any alternatives $c, d \notin \{a, b\}$, finishing the proof. ⊠

This proves the proposition, and ergo our theorem!

Lecture 2

The Borsuk-Ulam Theorem (9/15/20)

Today's speaker is Hannah Turner, a 6th year Ph.D student. We'll be talking about the Borsuk Ulam Theorem!

2.1 Continuous Maps

We talk about maps from n-dimensional spheres to $\mathbb{R}^n$. Usually we talk about maps $f : \mathbb{R} \to \mathbb{R}$ that are continuous, "don't lift your pencil". In topology, preimage of open sets are open, AKA for $f : X \to Y$, points are close in Y imply sets are close in X. For the scope of this talk, assume topological spaces are metrizable.

Definition 2.1 (Sphere)**.** We have $\mathbb{R}^n = (x_1, x_2, \cdots, x_n)$ for $x_i \in \mathbb{R}$. We define the *sphere* notated S^{n-1} as the set

$$\{x_i \,\big|\, |x_i| = 1\},$$

or the set of points that are a distance 1 from the origin. For example, $S^1 \subseteq \mathbb{R}^2$, $S^2 \subseteq \mathbb{R}^3$.

Let talk about maps $S^1 \to \mathbb{R}$. Deform the circle into squiggly things then smash it. Or you can turn it into a square then squish it. Yay for deformation retractions! Also: S^1 is compact, so it maps onto a closed and bounded interval. Note this map isn't onto.

2.2 The Borsuk-Ulam Theorem

Theorem 2.1 (Borsuk-Ulam). *Any map $f : S^n \to \mathbb{R}^n$ sends two antipodal points ($v \sim -v$) in S^n to the same point in $\mathbb{R}^n$.*

Example 2.1. Any map $S^1 \xrightarrow{f} \mathbb{R}$ sends two antipodal points in S^1 to the same point in $\mathbb{R}$. Look at $g(x) = f(x) - f(-x)$, where $g : S^1 \to \mathbb{R}$. Our new goal: show that $g(x)$ has a zero (this shows BU for $n = 1$). Pick our favorite point $x_0 \in S^1$, and assume $g(x_0) \neq 0$. So $g(x_0)$ is either positive or negative, that is $g(x_0) > 0$ or $g(x_0) < 0$.

Assume $g(x_0) > 0$: what happends to $-x_0$, the antipodal point?

$$g(-x_0) = f(-x_0) - f(-(-x_0)) = f(-x_0) - f(x_0) = -(f(x_0) - f(-x_0)) = -g(x_0).$$

The $g(-x_0) < 0$. Now we apply the IVT, but we have to be a little careful. For the usual $\mathbb{R} \xrightarrow{f} \mathbb{R}$, say $f(x) = 5$, $f(y) = 7$, we hit every value in between 5 and 7. What's important: S^1 is *path-connected* (so the IVT still applies, since f is a function from a path-connected space into $\mathbb{R}$). Then there exists some $x \in S^1$ such that $g(x) = 0$, finishing the example.

The proof in higher dimensions is more difficult. There are three flavors:

1. Algebraic Topology: Assign an algebraic invariant. Weird equation: $H_*(\mathbb{R}P^n_i \mathbb{F}_2)$
2. Combinatorics: Tucker's Lemma,
3. Set covering (Lusternik-Schnirelmann): For S^n, any $n+1$ open sets covering one of the sets must contain antipodal points (in at least one of the covering sets).

2.3 Corollaries of BU

Definition 2.2 (Homeomorphisms). A *homeomorphism* is a continuous function $f : X \to Y$ which has a continuous inverse $f^{-1} : Y \to X$, $f \circ f^{-1} = \mathrm{id}_X$.

Example 2.2. A map which is not injective cannot have an inverse! Because then one point would map to two, breaking the rules and causing society to fall into a complete collapse.

Example 2.3. Take the map from the half open interval to the circle, that is, $f : [0,1) \to S^1$. f is continuous, has an inverse, but the inverse isn't continuous. Intuition: points at the place where the "endpoints" are identified are now very far away in the preimage of the inverse. So f is a bijection but its inverse is not continuous, so f is NOT a homeomomorphism.

Corollary 2.1. *There is no homeomorphism from $S^n \to \mathbb{R}^n$. Any continuous function $f : S^n \to \mathbb{R}^n$ has $f(x) = f(-x)$, not even one to one!*

2.4 Pancakes!

Corollary 2.2 (Pancake Theorem). *Any two disks in the place can be cut exactly in half by one slice. This includes weirdly shaped disks! In general, if we have n amount of n-dimensional blobs, we would have an n-dimensional hyperplane (locally homeo to $\mathbb{R}^{n-1}$) in $\mathbb{R}^n$ that slices each n-dimensional blob exactly in half.*

Proof. Sketch of a proof: take our 3 objects A_1, A_2, A_3. Something about normal vectors and perpendicular planes. Measure the volume? (Measures??) Pick the plane that gives half of the sandwich. Repeat for every plane in the sphere, call each plane P_x (where half of the sandwich is on each side of any P_x). Define a map $f : S^2 \to \mathbb{R}^2$ by $x \mapsto (\mathrm{vol}(A_2)$ on the positive side of P_x, $\mathrm{vol}(A_3)$ on the positive side of $P_x)$. We know there are x_0 and $-x_0$ with $f(x_0) = f(-x_0)$ by BU. Man, I wish I could TEX figures in real time. So

$$x_0 \mapsto (\mathrm{vol}(A_2)P^+_{x_0}, \mathrm{vol}(A_3)P^+_{x_0}),$$
$$-x_0 \mapsto (\mathrm{vol}(A_2)P^+_{-x_0}, \mathrm{vol}(A_3)P^+_{-x_0}),$$

which are equal. The point is, we get the same plane but we're looking at it from two different directions, because $(\mathrm{vol}(A_2)P^+_{-x_0}, \mathrm{vol}(A_3)P^+_{-x_0}) = \left(\mathrm{vol}(A_2)P^-_{x_0}, \mathrm{vol}(A_3)P^-_{x_0}\right)$. $\mathrm{vol}(A_2)$ is cut in half by P_{x_0}, $\mathrm{vol}(A_3)$ is cut in half by P_{x_0}. ⊠

Lecture 3

What's the Putnam? (9/22/20)

Announcements: the reading groups are ready! We're studying analytic number theory, graph theory, and complexity theory. Also: social this Friday, Tiffs treats! Next week: Quantum computing, stay tuned.

Today's speaker is Dr. Rusin, an assistant professor here in the math department. He likes working with students that make their lives difficult for themselves (by doing hard problems). Some alternatives:

- The Bennett competition (only for calculus students). Problems that are not allowed to go on a final exam because they're hard. (I've read these on the walls before!) We also have linear algebra and differential equations exams.
- "Spy people" are mathematicians working for the NSA. In other words, traitors. There is some competition for math modeling that runs in February, in teams of three. It's the "anti-Putnam".

3.1 About the Putnam

Now let's talk about the Putnam: it's an annual math competition, open to undergraduates in the US and Canada (no more than 4, no bachelors). Mathematics only, once a year (historically, the first Saturday in December). It runs all day, from 9:00 to 3:00 in two groups of six questions. Reset your progress at the lunch break? `[Yes:No]`. College level topics: calculus, linear algebra, differential equations, number theory, topology, real analysis, abstract algebra, even statistics and mathematical physics. The questions are quite "accessible" on the surface: syke!

Some are about games: flashbacks to Fishman's Banach-Mazur game (Alice and Bob). It was actually me, linear algebra! Some practice strategies include working on old questions, learn the tricks. By tradition, the questions are arranged from easier to harder. So most people try the first question. The classic: the median score out of 120 is 1.

3.2 Problem A1 2019

Problem (2019 Putnam Question A1). *Determine all possible values of the expression*

$$A^3 + B^3 + C^3 - 3ABC,$$

where A, B, and C are nonnegative integers.

What do? Let's see... looks like number theory or linear algebra to me. Does this remind me of a group I know? Find a pattern, generalize the pattern, determine the relation, write a proof. If $A = B = C$, then $3A^3 - 3AAA = 0$, taking care of the trivial case. I hate how Zoom kills my battery.

By FLT?? Complete madlad. Unfortunately it's not relevant (very sad). I wish I could see more examples of proof by overkill. What if we have the numbers of the form $A-1, A, A+1$? Expanding the cubes, we get $3A^3 + 6A - 3ABC = 3A^3 + 6A - 3A(A^2-1) = 9A$. So we have all multiples of 9 at the least. Is the output all integers, and we just show it this way? Seems easier than classifying stuff in the domain (close integers, etc). Similarly, for $A, A, A+1$, we have

$$\sum = A^3 + A^3 + A^3 + 3A^2 + 3A + 1 - 3A^2(A+1) = 3A^3 - 3A^3 + 3A^2 - 3A^2 + 3A + 1 = 3A + 1.$$

Eventually, you keep plugging stuff in but you can't find a solution set with some and some, anything congruent to 3 or 6 (mod 9). Then the fact that the domain is non-negative, that messes with the output formulas, eventually only non-negative outputs. This follows from the AM-GM inequality (oldest trick in the book).

We're not done yet: factor the polynomial, plug it into a matrix. There's a connection with something called a *circulant* matrix, do stuff with the eigen-whatever. Not all solutions have to be elegant, just solve them. Let's look at the winners: all from MIT, great. But we got honorable mention yay!

This year, the tentative date for the Putnam is February 20, 2021. If everyone's back and running on campus, they intend to hold the competition as usual. Backup plan: they're still going to run the competition, but no prizes and no winners. Maybe hybrid too. Look at the web pages at the math department (Dr. Rusin's website) for the Zoom link. Hook' em!

Digest this problem in your free time: you can prove it in two words. ???

Problem. *Given a lattice grid, you can make triangles with the vertices as points. Is there an equilateral triangle with integer coordinates?*

Lecture 4

Solving the word problem with geometry (10/13/20)

Today's speaker is Teddy Weisman, a 5^{th} year Ph.D student. We'll be talking about regular expression in computer science and their connection to geometry (connecting math and linux!) JK, I was baited :(((I really like regular expressions this is sad

Favorite thing to do in math: see a weird abstract problem, look at it hard enough, it turns into geometry.

Here's a simple puzzle, take an alphabet, write a word (finite seq of letters), basically group presentations. We have the free group on four letters with relations $\langle a, b, c, d \mid [a, b], [a, c], [b, d] \rangle$: wait, you can replace the commutators $[a, c]$ and $[b, d]$ with the empty word $\{\}$, but not $[a, b]$. Goal: is every word trivial?

Example 4.1. $abcd \to bacd \to bd \to \{\}$, so that's trivial. $bacbd \to bbd \to b$, non trivial.

Definition 4.1. Two words are **equivalent** if you can get from one to the other with finitely many replacements.

4.1 The word problem

Question: Can you write an algorithm that determines if any word is equivalent to the empty word?

Answer: Yes, there is one. Rewrite our generating set as a, b, a^{-1}, b^{-1}, $ab = ba$, $aa^{-1} = a^{-1}a = \{\}$, $bb^{-1} = b^{-1}b = \{\}$. This turns into algebra, take for example the word $aba^{-1}bbb^{-1}aa^{-1}ba$, we can switch around and simplify stuff. Furthermore, $ab^{-1} = b^{-1}a$ and $a^{-1}b = ba^{-1}$. How do we check if it's the empty word? Count the number of a's, subtract the number of a^{-1}'s, for example the word from earlier is now b^3a which is nontrivial.

Definition 4.2. A **string rewriting system** S is a pair $S = (A, R)$, where A is a finite set of letters and R is a finite set of pairs of words in A. If each letter $a \in A$ has an inverse $a^{-1} \in A$, with a replacement rule $a^{-1}a = aa^{-1} = \{\}$, then S is a group presentation.

Question: Can you find an algorithm with takes as input a string writing system S and a word in an alphabet for S, which determines if the word is equivalent to 1 (trivial)? (I think the answer is no...)

The answer is no! I got it right yay. Computers are not capable of doing this. No wonder the group presentations from van Kampens are so nasty. This is equivalent to the halting problem.

You can naïvely try a *brute force algorithm* :

- Set $n = 1$.

- Apply all possible sequences of n subsitutions to the original word.
- If any are the empty word, we're done.
- Otherwise, increment n and try again.

However, if a word is nontrivial, this algorithm doesn't halt. It would work if: we know the max number (finite) of subsitutions we need to try, just try everything. The catch is we need to know this number, which we don't always know.

4.2 Applying geometry

Let's go back to the original string rewriting system, with $A = \{a, a^{-1}, b, b^{-1}\}$ and $R = \{[a, b], aa^{-1}, a^{-1}a, bb^{-1}, b^{-1}b\}$. Mapping works to (x, y) pairs: consider

$$\{\text{words in } A\} \longrightarrow \{(x, y) \text{ pairs}\},$$

where $w \mapsto (x, y)$, $x =$ number of a's minus number of a^{-1}'s, and $y =$ number of b's minus the number of b^{-1}'s. There's a nice correspondence between equivalence classes of words to (x, y) pairs. Equivalent words map to some (x, y) pair, and inequivalent words map to different (x, y) pairs. Unfortunately, now he's gonna draw a picture which I don't have the skills to do in real time yet. Think of the (x, y) pairs as a grid, and draw some lines. Let $(0, 0) = 1$, $(0, 1) = a$, $(1, 0) = b, (1, 1) = ab, (-1, 0) = a^{-1}, (-1, 1) = a^{-1}b$ and so on. In this graph, the vertex set $V \mapsto \{\text{equivalence classes of words}\}$, and the edge set $E \mapsto \{\text{words}\}$. For example, for a word $aabba^{-1}a^{-1}b^{-1}$, draw a path starting at a and you'll end up at b (are these Cayley graphs??) Loops are special paths that correspond to trivial words.

We can think about operations geometrically: cancelling an inverse means getting rid of backtracking. Another operation is given by the commutator relation: it turns out this is the same thing as adding or deleting loops at a point on the path. For example, for a word $aab^{-1}b^{-1}a^{-1}a^{-1}bb$, which is a loop. Apply the commutator relation, then this word maps to $aab^{-1}b^{-1}a^{-1}[aba^{-1}b^{-1}]a^{-1}bb$, this adds a unit loop at $(1, -2)$. However, we can cancel inverses, and then examining the interior it looks like we deleted the aforementioned loop.

Idea: adding a substitution to a word changes the area enclosed by at most 1. Good thing is, if our loop encloses an area V, then we need to apply at most V substitutions to make it trivial. This is what we wanted for the brute force algorithm to work, the largest number of substitutions we can make. **Question:** How much area can a loop of length L in $\mathbb{R}^2$ enclose? This problem is called *Dido's problem*, in which she was told to enclose the land she was to receive with a finite string (she somehow managed to get all of Carthage?) We have $A = \frac{L^2}{4\pi}$ for an arc of length L (not entirely sure). Given a word of length L:

- Apply a substitution of the form $aba^{-1}b^{-1} = 1$,
- Get ride of backtracking,
- Check if the word is 1,
- Keep going until we've applied L^2 subsitutions,
- If the word isn't 1, start over with a different sequence.

The idea is, there's only a finite amount of words with length L^2, so this algorithm will work.

4.3 Cayley graphs

Here we are: Given a group presentation (A, R), where the vertices are just the equivalence classes of words in A. $v_1 \sim v_2$ if a word representing v_2 is w_1a, where w_1 represents v_1 and $a \in A$. Basically, edges on the Cayley graph represent words and points are equivalence classes. We can also look at the Cayley graph of $F_2 = \mathbb{Z} * \mathbb{Z}$, the free group on two letters. Regular graph: every vertex has the same degree, in this case every vertex can serve as the basepoint of the graph. Basically, F_2 is the universal cover of $S^1 \vee S^1$.

Example 4.2. Complicated example: $A = \{a, b, c, d, e\}$, $R = \{aa = bb = cc = dd = ee\}$ plus a lot of other relations. Then this is a tiling with five things of 90°, doesn't work in Euclidian space but it works in non-Euclidian space. How does this help with solving the word problem? Look at loops in hyperbolic space.

Question: How much area can we enclose with a loop of length L is the hyperbolic plane? This time, of length L, Area $\leq 3L + 20$, need a long string to enclose less area. So this is a linear isometric inequality[2]. This solves the word problem! Go through $3L + 20$ relations, if we aren't dont, it's not the identity.

Lecture 5

Knot Groups and Bi-orderability (10/13/20)

Today's speaker is Jonathan Johnson, a 4[th] year Ph.D student. Whoops, I came thirty minutes late into the talk. Good thing in Zoom, doors don't slam so the thing where everybody awkwardly stares at you for a sec doesn't happen.

I feel like we talked about group presentations last talk (the last time I was here, about geometry and group theory), hmm....

Example 5.1. We have $\langle a, b \mid [a, b]\rangle \simeq \mathbb{Z} \oplus \mathbb{Z}$. Also, note that $\langle x_1, \cdots, x_n\rangle$ is the **free group** on n-letters[3] (of rank n).

5.1 Bi-orderability

Definition 5.1 (Bi-order)**.** A **bi-order** of a group is a total order of the groups elements which is invariant under both left and right multiplication. That is, for all $a, b, g \in G$, if $a < b$ then $ga < gb$, $ag < bg$. The term "bi" comes from the fact that it works with both left and right multplication (people spend a lot of time studying just right multiplication).

A group is said to be **bi-orderable** if it admits a bi-order.

Example 5.2. Our favorite groups $\mathbb{Z}, \mathbb{Q}$, and $\mathbb{R}$ are BO (bi-orderable). Now consider $\mathbb{Z}/4$. If $\mathbb{Z}/4$ were to be BO, then $0 < 1 < 2 < 3 < 0$, a contradiction.

Proposition 5.1. *If a group has a non-trivial torsion element, then it isn't BO. By torsion element, we mean that for $g \in G$, g is a **torsion element** if $g \neq 1$ and $g^k = 1$ for some $k \in \mathbb{Z}^+$. In other words, g has* finite order.

As an example, take the group $G = \langle a, b \mid aba^{-1} = b^{-1}\rangle$. Assume $1 < b \implies b^{-1} < 1$. Then $1 = a_1a^{-1} < aba^{-1} = b^{-1}$, a contradiction. So this group isn't bi-orderable either.

5.2 Knots

Let's talk about knots.

Definition 5.2 (Knot)**.** A **knot** is a (smooth) simple closed curve in $\mathbb{R}^3$. Two knots are equivalent if one can be **isotoped** to the other. A **knot diagram** is a projection of a knot of a knot onto a plane with no triple intersections and remembering crossing information. AAAAaaah Dehn presentations!!!

Example 5.3. No pictures, sadly. We can talk about knots like the trefoil knot, the unknot (S^1), cool looking twist (pretzel knot), and the figure-eight knot.

Turns out given a knot K, we can get a group from it, called the **knot group** . AAAAAAAAAAAAAAAA wïrtïnger presentations

Steps:

[2] I don't think I got the long words right...

[3] OK, I know this is not what you're supposed to call it. Just let me have my fun!

1. Draw an oriented diagram of K.
2. Use a generator for each overstrand.
3. For each crossing, write a relator according to the rule below: Wirtinger I think...

Example 5.4. For example, $\pi(\mathsf{Tref}) \simeq \langle a, b, c \mid ab = bc, ca = ab, bc = ca\rangle$.

Question: When are knot groups bi-orderable? Answer: This turns out to be quite hard.

Recall $\pi(\mathsf{Tref})$, let's try to make this group a bit simpler. We can eliminate a generator by noting that $c = aba^{-1}$, so our new presentation is given by $\langle a, b \mid ab = baba^{-1}, baba^{-1} = ab\rangle$. Both of these relations are saying that $aba = bab$, so our new nice presentation is finalized by

$$\pi(\mathsf{Tref}) = \langle a, b \mid aba = bab\rangle.$$

So a and b don't commute, and $a \neq bab^{-1}$. Suppose $a < bab^{-1}$, and so $a = aaa^{-1} < abab^{-1}a^{-1}$ (we can arrange them weirdly cuz of bi-orderability). On the right side, stuff cancels out and we get b, so $a < b$. But then $b^{-1} < a^{-1}$, so

$$ab^{-1}a^{-1} < aa^{-1}a^{-1}$$
$$ab^{-1}a^{-1} < a^{-1}$$
$$bab^{-1}a^{-1}b^{-1} < ba^{-1}b^{-1}.$$

Return to our old relation $aba = bab$, so $bab^{-1}a^{-1}b^{-1} = a^{-1}$. Some more relation magic happens, then $a^{-1} > bab^{-1}$, a contradiction. So $\pi(\mathsf{Tref})$ doesn't admit a bi-order.

Theorem 5.1 (Perran-Roltsen)**.** *$\pi(\mathsf{Fig\,8})$ is bi-orderable.*

No proof, but a general idea is that we can write

$$\pi(\mathsf{Fig\,8}) = \langle a, b, t \mid tat^{-1} = b^{-1}, tbt^{-1} = ba\rangle.$$

Some relation magic happens, then $t^{-1}at = ab, t^{-1}bt = a^{-1}$. Suppose $w \in \langle a, b\rangle$, then conjugation by t, twt^{-1} and $t^{-1}wt$ are in $< a, b >$. This group ends up being $\langle a, b\rangle \rtimes \langle t\rangle$ given by those relations (semidirect product). What does this mean? Every element of $\pi(\mathsf{Fig\,8})$ is of the from $t^n w$ where $n \in \mathbb{Z}$ and $w \in \langle a, b\rangle$. This conjugation here is really defining an isomorphism, say $\varphi_t\colon \langle a, b\rangle \to \langle a, b\rangle$, given by $\varphi_t(w) = twt^{-1}$. Here's the fun fact that makes it all worth it.

Proposition 5.2. *There is a bi-order on $\langle a, b\rangle$ which is invariant under φ_t.*

By this proposition, we can bi-order the knot group of the figure eight. Say $g_1, g_2 \in \pi(\mathsf{Fig\,8})$. Then $g_1 = t_1^n w_1$, $g_2 = t_2^n w_2$ for $n_1, n_2 \in \mathbb{Z}$, $w_1, w_2 \in \langle a, b\rangle$. Then $g_1 < g_2$ if and only if $n_1 < n_2$ (or $n_1 = n_2$ and $w_1 <_F w_2$), where $<_F$ is the supposed bi-order on $\pi_1(\mathsf{Fig\,8})$. Note that the theorem actually showed this for a whole class of groups, as opposed to just the figure eight.

Lecture 6

4-Manifolds (10/13/20)

Today's speaker is Kai Nakamura, a 3rd(?) year Ph.D student.

Definition 6.1 (Manifolds)**.** A **topological manifold** is a space that locally looks like $\mathbb{R}^n$ (locally Euclidian), and some other stuff too (Hausdorff, charts, etc). For example, S^n. A **smooth structure** on a topological manifold is essentially a way to do calculus on such space. For example, we do multivariable calculus on $\mathbb{R}^n$ with the standard smooth structure. So a **smooth manifold** is a topological manifold with a smooth structure.

We want to study these manifolds up to some notion of equivalence, so we study the manifolds up to **homeomorphism** (continuous map with continuous inverse) and **diffeomorphism** (differentiable map with differentiable inverse).

Example 6.1. Topologists say that S^1 and the ellipse are the same thing, because we can find a homeomorphisms. Similarly, $\mathbb{R}$ with the smooth structure and the open interval $(0, 1)$.

6.1 Classifying manifolds

In the 2-dimensional case, this is the classification of surfaces. The one dimensional case is pretty trivial, for example a line, open, half open, closed interval etc. Some examples of surfaces include $S^2, \mathbb{T}$, and the genus n-surface. These are also all smooth.

Classification of Surfaces. *Every closed orientable surface is homeomorphic or diffeomorphic to one of the examples we mentioned above.*

Something fun to think about: torus with a hole in it. In three dimensions, the most famous result is probably Poincare's conjecture (no longer a conjecture), which asks whether a simply-connected closed 3-manifold homeomorphic to a sphere. This is true! And Grigory Perelman ran off into the woods and was never to be seen again. Simply connected means π_1 is trivial.

In dimension 7, Milnor (1956) constructed smooth manifolds M that are homeomorphic to S^7 but not diffeomorphic. This shocked a lot of people, because in dimensions two or three we automatically get a smooth structure.

Smale (1961) proved the topological Poincare conjecture for all dimensions greater than or equal to 5, that is, a smooth n-manifold homotopy equivalent to S^n is homeomorphic to S^n. Although the theorem gives a homeomorphism, it actually uses smooth techniques. If you interset two spheres you can try to find a boundary circle (then rotate it?). This works for dimensions 5 and higher, and is known as **Whitney's trick**. However, this doesn't work for 4-manifolds.

Freedman (1981) showed that this worked topologically, which resulted in a classification of topologically simply connected closed 4-manifolds, also showing the topological 4-dimensional Poincare conjecture. This worked topologically, so maybe this works smoothly too?

Donaldson (1982) analyzed the anti-self dual Yang-Mills equations on a smooth 4-manifold to prove his famous Diagonalization theorem. The takeaway is that it showed the Whitney trick can't work smoothly in dimension four, unfortunately.

6.2 Exotic $\mathbb{R}^4$

These are examples of manifolds homeomorphic to $\mathbb{R}^4$, but not diffeomorphic to $\mathbb{R}^4$. In other words, there are different ways to do calculus on $\mathbb{R}^4$. This is a huge shock, because it doesn't happen in other dimensions, since smooth manifolds homeomorphic to $\mathbb{R}^n \implies$ diffeomorphic to $\mathbb{R}^n$ if $n \neq 4$. In fact, there are uncountably many distinct exotic $\mathbb{R}^4$. When you get to universal exotic $\mathbb{R}^4$ it gets very strange, since you have uncountably many submanifolds that aren't homeomorphic to each other. Furthermore we have small exotic $\mathbb{R}^4$, which are smooth manifolds of $\mathbb{R}^4$-std that are exotic $\mathbb{R}^4$.

6.3 The Conway knot is (k)not slice

Say we have the Conway knot C, then it has a sibling (mutation) the Kinoshita-Terasaka knot, by flipping a region around.

Definition 6.2 (Knot)**.** A **knot** K is an embedding $S^1 \hookrightarrow S^3$, where $S^3 = \partial B^4$, $D^2 \hookrightarrow B^4$, $\partial D^2 \hookrightarrow K \subseteq B^4$. If the disk above doesn't intersect itself as an embedding, we say the knot is **slice**.

The Kinoshita-Terasaka knot is slice. For a long time, it was an open problem whether the Conway knot was slice. Usually mutants provided a pretty good picture, but it didn't work, and neither did a bunch of invariants. For example $T(C) = 0, S(C) = 0$, nobody could figure it out. For a while, this was the last knot with 11 crossing that we hadn't classified it yet.

Lisa Piccirillo's Proof. Consider the disk embedded in S^4, with an equatorial S^3. If it's slice, it must bound a disk here. If the knot is slice, we can thicken up the disk, and take the union of B^4 (hemisphere) and union it with the thickened slice disk. This is called the **knot trace** $X(K)$. A knot K is slice iff $X(K) \subseteq S^4$. Now asking whether C is slice is the same thing as asking whether $X(C)$ embeds in S^4. Piccirillo's insight was to use another knot D such that $X(C) \cong X(D)$ a diffeomorphism. Now is D slice? $S(D) = 2 \implies D$ is not slice, so $X(D)$ doesn't embed in S^4 and so $X(C)$ doesn't embed in S^4, therefore C is not slice.