The Yoneda Lemma

A brief introduction to category theory

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Categories

Definition

A **category** $\mathcal{C}$ consists of the following data:

- A set of objects $\text{Ob}(\mathcal{C})$,
- A set of morphisms $\text{Hom}(A, B)$ for any two objects $A, B \in \text{Ob}(\mathcal{C})$,

such that for any two morphisms $f : A \to B, g : B \to C$ there exists a morphism $g \circ f : A \to C$. This data is subject to the following rules:

- Composition satisfies associativity,
- There exists an identity morphism $\text{id}_A : A \to A$ for any object $A \in \mathcal{C}$ satisfying $f \circ \text{id}_A = \text{id}_B \circ f$ for any $A \xrightarrow{f} B$.

Examples

Example

Some typical examples of categories:

- Sets
- Groups
- Finite dimensional vector spaces
- Topological spaces

Example

Categories with one object can be viewed as monoids, and if every morphism is invertible then they become groups.

Functors

Definition

A (covariant) **functor** $K: \mathcal{C} \to \mathcal{D}$ between two categories $\mathcal{C}, \mathcal{D}$ associates to each object of $\mathcal{C}$ an object of $\mathcal{D}$, and to each morphism $[A \xrightarrow{f} B] \in \mathcal{C}$ a morphism $[K(A) \xrightarrow{Kf} K(B)] \in \mathcal{D}$. Furthermore, $K(\mathrm{id}_A) = \mathrm{id}_{K(A)}$ and $K(A \xrightarrow{f} B \xrightarrow{g} C) = [K(A) \xrightarrow{Kf} K(B) \xrightarrow{Kg} K(C)]$ for $A, B, C \in \mathcal{C}$.

Example

Some examples of covariant functors:

- The functor $\mathrm{Grp} \to \mathrm{Set}$ assigning groups to their base set
- $\mathrm{Set} \to \mathrm{Grp}$ where $A \in \mathrm{Set}$ gets sent to the free group on A is a functor
- $\pi_1 : \mathrm{Top}_* \to \mathrm{Grp}$ is a functor because the induced homomorphism preserves composition and identities

Covariant or contravariant?

Definition

The **opposite category** $\mathcal{C}^{\mathrm{op}}$ of a category $\mathcal{C}$ is the same as $\mathcal{C}$ except the direction of the arrows is reversed (every $[A \xrightarrow{f} B] \in \mathcal{C}$ corresponds to a $[B \xrightarrow{f^{\mathrm{op}}} A] \in \mathcal{C}^{\mathrm{op}}$).

Definition

A **contravariant functor** is a covariant functor $K\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$. We could also say a contravariant functor is a covariant functor which switches the direction of the arrows on composition, or $K(A \xrightarrow{f} B \xrightarrow{g} C) = [K(C) \xrightarrow{Kg} K(B) \xrightarrow{Kf} K(A)]$.

Example

The dual space functor $\mathrm{Vect}_k^{\mathrm{op}} \to \mathrm{Vect}_k$ sending
$V \mapsto V^* = \mathrm{Hom}(V, k)$,
$[V_1 \xrightarrow{T} V_2] \mapsto ([V_2 \to k] \mapsto [V_1 \xrightarrow{T} V_2 \to k])$ is contravariant.

Natural transformations

Definition

Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be functors. A **natural transformation** $\alpha\colon F \to G$ associates to any object $A \in \mathcal{C}$ a morphism $F(A) \xrightarrow{\alpha_A} G(A)$ in $\mathcal{D}$, satisfying the naturality diagram below for any $f\colon A \to B$.

$$\begin{array}{ccc} F(A) & \xrightarrow{Ff} & F(B) \\ \downarrow{\scriptstyle \alpha_A} & & \downarrow{\scriptstyle \alpha_B} \\ G(A) & \xrightarrow{Gf} & G(B) \end{array}$$

Some examples

Example (The Yoneda functor)

For a category $\mathcal{C}$ with $A, X, Y \in \mathcal{C}$, $\mathrm{Hom}(-, A)\colon \mathcal{C} \to \mathsf{Set}$ defined by $X \mapsto \mathrm{Hom}(X, A), [X \xrightarrow{f} Y] \mapsto ([Y \to A] \mapsto [X \xrightarrow{f} Y \to A])$ is a contravariant functor.

Example (Natural transformations)

- Consider the double dual functor $(-)^{**}\colon \mathsf{Vect}_k \to \mathsf{Vect}_k$, $V \to V^{**} = \mathrm{Hom}(\mathrm{Hom}(V, k), k)$. Let $[V \xrightarrow{f} k] \in V^*, v \in V$, then $\mathrm{eval}\colon (-)^{**} \to \mathrm{id}_{\mathsf{Vect}_k}$ is a natural transformation to the identity functor where $\mathrm{eval}_V\colon v \mapsto [f \mapsto f(v)]$.

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ \downarrow{\scriptstyle \mathrm{eval}_V} & & \downarrow{\scriptstyle \mathrm{eval}_W} \\ V^{**} & \xrightarrow{T^{**}} & W^{**} \end{array}$$

The Yoneda lemma

Lemma (Yoneda lemma)

If $K: \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ *is a contravariant functor and* $R \in \mathcal{C}$*, then there is a bijection of sets* $\mathrm{Nat}(\mathrm{Hom}(-, R), K) \simeq K(R)$*.*

Proof of the Yoneda lemma.

Let $\alpha\colon \mathrm{Hom}(-, R) \to K$ be a natural transformation. Then $\alpha_R\colon \mathrm{Hom}(R, R) \to K(R)$, particularly $\alpha_R(\mathrm{id}_R) \in K(R)$. Naturality tells us that for $D \xrightarrow{f} R$ the following diagram commutes

$$\begin{array}{ccc} \mathrm{Hom}(R, R) & \xrightarrow{\alpha_R} & K(R) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle Kf} \\ \mathrm{Hom}(D, R) & \xrightarrow{\alpha_D} & K(D) \end{array}$$

or in other words, $Kf(\alpha_R(\mathrm{id}_R)) = \alpha_D(f)$.

Proof of the Yoneda lemma (continued).

For $A \in \mathcal{C}$, let $\beta_A \colon \operatorname{Hom}(A, R) \to K(A)$ send $[A \xrightarrow{g} R] \mapsto Kg(b)$. Checking naturality, let $A, B \in \mathcal{C}$, $A \xrightarrow{f} B$.

$$\begin{array}{ccc} \operatorname{Hom}(A, r) & \xrightarrow{\beta_A} & K(A) \\ \uparrow f^* & & \uparrow Kf \\ \operatorname{Hom}(B, r) & \xrightarrow{\beta_B} & K(B) \end{array}$$

Let $B \xrightarrow{h} r \in \operatorname{Hom}(B, r)$, then

$$f^*h = [A \xrightarrow{f} B \xrightarrow{h} R] = [A \xrightarrow{h \circ f} R] \in \operatorname{Hom}(A, R).$$

So $\beta_A(h \circ f) = K(h \circ f)(b) = (Kf \circ Kg)(b)$ by contravariance.
Now $\beta_B(h) = Kh(b)$, so $Kf(\beta_B(h)) = Kf(Kh(b)) = (Kf \circ Kh)(b)$.
Therefore $\beta_A(f^*(h)) = Kf(\beta_B(h))$.

Proof of the Yoneda lemma (continued).

Let $b \in K(r)$, and $\beta\colon \operatorname{Hom}(-, r) \to K$, $\beta_A\colon A \xrightarrow{g} r \mapsto Kg(b)$. Then $\beta_r(\mathrm{id}_r) = K(\mathrm{id}_r)(b) = \mathrm{id}_{\mathsf{Set}}(b) = b$. Let $\alpha\colon \operatorname{Hom}(-, r) \to K$ be a natural transformation, and consider $\alpha_r(\mathrm{id}_r) \in K(r)$. Then associate to this the natural transformation $\beta_A\colon A \xrightarrow{g} r \mapsto Kg(\alpha_r(\mathrm{id}_r))$.

$$\begin{array}{ccc} \operatorname{Hom}(r, r) & \xrightarrow{\alpha_r} & K(r) \\ \downarrow g^* & & \downarrow Kg \\ \operatorname{Hom}(A, r) & \xrightarrow{\alpha_A} & K(A) \end{array}$$

In the naturality diagram for α, note that $\beta_A(g) = (Kg \circ \alpha_r)(\mathrm{id}_r)$ is the result of following the red path. Also note that following the blue path gives $\alpha_A(g^*(\mathrm{id}_r)) = \alpha_A(g)$, so $\beta_A(g) = \alpha_A(g)$. Then $\beta = \alpha$, and we are done. $\boxtimes$