# Notes on de Rham Cohomology

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I'm excited to say that I'm participating in the Directed Reading Program (DRP) this semester, mentored by Arun Debray! (Read more here: web.ma.utexas.edu/users/drp). This semester, I'm following a book called *From Calculus to Cohomology: De Rham cohomology and characteristics classes* by Madsen and Tornehave.

These are the full version of the notes, taken to help me learn the material. I plan on summarizing my results in a sort of exposition style to put on the DRP website, as well as a beamer presentation for the symposium. I plan on having all three files hosted on my website somewhere, probably around here: https://git.simonxiang.xyz/math_notes/files.html

### Prerequisites

The reader should be familiar with multivariable calculus and linear algebra at the minimum, as well as basic group theory (up to the first isomorphism theorem). Some things that are helpful but not necessary include:

- Basic analysis, including open and closed sets, and the inverse function theorem.
- Point-set topology would be very nice.
- Algebraic topology would be very helpful, but I assume no knowledge of cohomology.

In general, these notes will be taken like you know what open sets are, properties of connected spaces, what a commutative diagram is, stuff like that (because they were taken to help me learn the material). But for the condensed paper, I plan on introducing everything I need (besides stuff from calculus and linear algebra), so they can be somewhat self contained.

## Contents

Lecture 1

# Motivation

## 1.1 Calculus

**Question.** Let $f : U \to \mathbb{R}^2$ be a smooth function, where $U \subseteq \mathbb{R}^2$ is open. Is there a smooth function $F : U \to \mathbb{R}$ such that $\partial_{x_1} F = f_1,\ \partial_{x_2} F = f_2$, where $f = (f_1, f_2)$? Note that this implies $\partial_{x_2} f_1 = \partial_{x_1} f_2$. Is this a sufficient condition to show the existence of $F$?

**Example 1.1.** Consider $f : \mathbb{R}^2 \to \mathbb{R}^2$, where

$$f(x_1, x_2) = \left( \frac{-x_2}{x_1^2 + x_2^2}, \frac{x_1}{x_1^2 + x_2^2} \right)$$

Now

$$\partial_{x_2} f_1 = \frac{-(x_1^2 + x_2^2) + 2x_2^2}{(x_1^2 + x_2^2)^2} = \frac{x_2^2 - x_1^2}{(x_1^2 + x_2^2)^2},$$
$$\partial_{x_1} f_2 = \frac{(x_1^2 + x_2^2) - 2x_1^2}{(x_1^2 + x_2^2)^2} = \frac{x_2^2 - x_1^2}{(x_1^2 + x_2^2)^2}.$$

So $f$ satisfies $\partial_{x_2} f_1 = \partial_{x_1} f_2$. However, we have no $F : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$: assume there was such an $F$, then

$$\int_0^{2\pi} \frac{d}{d\theta} F(\cos\theta, \sin\theta)\, d\theta = F(1,0) - F(1,0) = 0.$$

But
$$\frac{d}{d\theta} F(\cos\theta, \sin\theta) = \frac{dF}{dx}(-\sin\theta) + \frac{\partial F}{\partial y}\cos\theta = -f_1(\cos\theta, \sin\theta)\sin\theta + f_2(\cos\theta, \sin\theta)\cos\theta = 1$$

by the chain rule, a contradiction. So we have procured a counterexample.

**Definition 1.1** (Star-shaped)**.** A subset $X \subseteq \mathbb{R}^n$ is **star-shaped** with respect to $x_0 \in X$ if the line segment $\{t x_0 + (1-t)x \mid t \in [0,1]\}$ is contained in $X$ for all $x \in X$.

**Theorem 1.1.** *Let $U \subseteq \mathbb{R}^2$ be open and star-shaped. Then for any smooth function $(f_1, f_2) \colon U \to \mathbb{R}^2$ satisfying $\partial_{x_2} f_1 = \partial_{x_1} f_2$, there exists a smooth function $F \colon U \to \mathbb{R}$ such that $\partial_{x_1} F = f_2,\ \partial_{x_2} F = f_1$.*

*Proof.* Messy. ⊠

## 1.2 Sneak peek of cohomology

Say $U \subseteq R^2$ is open, then let $C^\infty(U, \mathbb{R}^k)$ be the vector space of smooth functions $\phi : U \to \mathbb{R}^k$. Define the **gradient** and **curl** functions[1] $\operatorname{grad} \colon C^\infty(U, \mathbb{R}) \to C^\infty(U, \mathbb{R}^2),\ \operatorname{curl} \colon C^\infty(U, \mathbb{R}^2) \to C^\infty(U, \mathbb{R})$ by

$$\operatorname{grad}(\phi) = \left( \frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2} \right), \qquad \operatorname{curl}(\phi_1, \phi_2) = \frac{\partial \phi_1}{\partial x_2} - \frac{\partial \phi_2}{\partial x_1}.$$

Note that the curl of the gradient is zero, or $\operatorname{curl} \circ \operatorname{grad} = 0$. So the kernel of the curl contains the image of the gradient, since mapping im(grad) by curl gives zero. Since curl and grad are linear, both ker(curl) and im(grad) are (infinite-dimensional) vector spaces, furthermore, im(grad) is a subspace of ker(curl). So we can consider the quotient space (since vector spaces are abelian groups) $H_1(U) = \ker(\operatorname{curl}) / \operatorname{im}(\operatorname{grad})$. This is a sneak peek of the *cohomology* groups (in this case, vector spaces) assigned to a space. Somehow the cohomology groups tend to be finite-dimensional.

[1] The book uses *rotation* instead of curl, but I think this is the standard notation.

$$
\begin{tikzcd}
\overset{\ker(\mathrm{grad})}{H^0(U)} \arrow[r] & \overset{\ker(\mathrm{curl})/\operatorname{im}(\mathrm{grad})}{H^1(U) = 0} \arrow[r] & H^2(U) \\
C^\infty(U,\mathbb{R}) \arrow[u] \arrow[r, "\mathrm{grad}"] & \underset{\text{exact}}{C^\infty(U,\mathbb{R}^2)} \arrow[u] \arrow[r, "\mathrm{curl}"] & C^\infty(U,\mathbb{R}) \arrow[u]
\end{tikzcd}
$$

Figure 1: The commutative diagram of gradient and curl for $U$ star-shaped.

Now Theorem 1.1 becomes the statement "$H^1(U) = 0$ whenever $U \subseteq \mathbb{R}^2$ is star-shaped". To see this, note that ker(curl) consists of precisely the functions $\phi\colon U \to \mathbb{R}^2$ such that $\partial_{x_2}\phi_1 = \partial_{x_1}\phi_2$, and if the image of grad are such functions $\phi$ (since $\ker(\mathrm{curl}) = \operatorname{im}(\mathrm{grad})$), then there must exist an $F \in C^\infty(U,\mathbb{R})$ mapping onto $\phi = (f_1, f_2)$, where $\partial_{x_1} F = f_2$, $\partial_{x_2} = f_1$.

We know that $H^1(\mathbb{R}^2 \setminus \{0\}) \neq 0$, since Example 1.1 details a function in ker(curl) that doesn't get mapped onto by im(grad). We will see later that $H^1(\mathbb{R}^2 \setminus \{0\})$ is 1-dimensional as a vector space, and that $H^1(\mathbb{R}^2 \setminus \bigcup_{i=1}^k \{x_i\}) \cong \mathbb{R}^k$. So the dimension of the cohomology groups gives us data about how many "holes" a space has. We will introduce cochain complexes and coboundaries later, but for now let us define $H^0(U) = \ker(\mathrm{grad})$ analagously. This is well-defined for open sets $U \subseteq \mathbb{R}^k$ for $k \geq 1$, for

$$\mathrm{grad}(f) = \left(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}\right).$$

**Theorem 1.2.** *An open set $U \subseteq \mathbb{R}^k$ is connected iff $H^0(U) = \mathbb{R}$.*

*Proof.* If $f \in \ker(\mathrm{grad})$ (so $\mathrm{grad}(f) = 0$), then $f$ is locally constant, that is, every $x_0 \in U$ has a neighborhood $V(x_0)$ such that $f(x) = f(x_0)$ for $x \in V(x_0)$. This makes sense because having zero derivative geometrically means "zero rate of change", so the function will be constant if we "zoom in close enough". To see this, apply the mean value theorem to the closure of a neighborhood around $x_0$, say $[a,b] \subseteq U$. Then $f'(c) = \frac{f(b)-f(a)}{b-a}$, and since $f'(c) = 0$, $f(b) - f(a)$. Since the derivative is zero everywhere, this implies the image of the neighborhood (and then $x_0$) is constant. This generalizes to multiple variables by parametrizing by one variable.

Now suppose $U$ is connected. Then locally constant functions are actually constant, since for $x_0 \in U$, the set

$$\{x \in U \mid f(x) = f(x_0)\} = f^{-1}(f(x_0))$$

is closed since it's the preimage of a closed set by the continuity of $f$, and open since $f$ is locally constant (every neightborhood has apoint). So since this set is nonempty, by connectedness this is all of $U$, and $H^0(U) = \mathbb{R}$.

Conversely, if $U$ is not connected, then we have a smooth, surjective function $f : U \to \{0,1\}$ defined by taking all but one of the connected components to 0, and the other to 1. Since $f$ is locally constant, $\mathrm{grad}(f) = 0$, so $\dim H^0(U) > 1$. We can easily extend this to show $\dim H^0(U) > 1$ by replacing $\{0,1\}$ with $\{1, \cdots, n\}$, where $n$ is the number of connected components of $U$. ⊠

Now let's move on to functions of three variables. Let $U \subseteq \mathbb{R}^3$ be open. Define

$$
\begin{aligned}
\mathrm{grad}\colon C^\infty(U,\mathbb{R}) \to C^\infty(U,\mathbb{R}^3), &\quad f \mapsto \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right), \\
\mathrm{curl}\colon C^\infty(U,\mathbb{R}^3) \to C^\infty(U,\mathbb{R}^3), &\quad (f_1,f_2,f_3) \mapsto \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial f_3}, \frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}, \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right), \\
\mathrm{div}\colon C^\infty(U,\mathbb{R}^3) \to C^\infty(U,\mathbb{R}), &\quad (f_1,f_2,f_3) \mapsto \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}.
\end{aligned}
$$

Note that $\mathrm{curl} \circ \mathrm{grad} = 0$, and $\mathrm{div} \circ \mathrm{curl} = 0$. Most textbooks leave this as an exercise but let's work this out in detail.

$$
\begin{gathered}
\mathrm{curl}\left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right) = \left(\frac{\partial^2 f}{\partial x_2 \partial x_3} - \frac{\partial^2 f}{\partial x_3 \partial x_2}, \frac{\partial^2 f}{\partial x_3 \partial x_1} - \frac{\partial^2 f}{\partial x_1 \partial x_3}, \frac{\partial^2 f}{\partial x_1 \partial x_2} - \frac{\partial^2 f}{\partial x_2 \partial x_1}\right) = 0, \\
\mathrm{div}\left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}, \frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}, \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right) = 0.
\end{gathered}
$$

The first equality is true because mixed partial derivatives commute, and the second because the first component in the expression for curl has no part containing $x_1$. So $\frac{\partial}{\partial x_1}\left(\frac{\partial f_3}{\partial x_2}-\frac{\partial f_2}{\partial x_3}\right)=\frac{\partial^2 f_3}{\partial x_1 \partial x_2}-\frac{\partial^2 f_2}{\partial x_1 \partial x_3}=0$, and so on.

Define $H^0(U)$, $H^1(U)$ as earlier and set $H^2(U)=\ker(\operatorname{div})/\operatorname{im}(\operatorname{curl})$.

**Theorem 1.3.** *For an open star-shaped set in $\mathbb{R}^3$ we have $H^0(U)=\mathbb{R}$, $H^1(U)=0$, and $H^2(U)=0$.*

$$\begin{tikzcd}
\overset{\ker(\mathrm{grad})}{H^0(U)} \arrow[r] & \overset{\ker(\mathrm{curl})/\operatorname{im}(\mathrm{grad})}{H^1(U)=0} \arrow[r] & \overset{\ker(\mathrm{div})/\operatorname{im}(\mathrm{curl})}{H^2(U)=0} \arrow[r] & H^3(U) \\
C^\infty(U,\mathbb{R}) \arrow[u] \arrow[r, "\mathrm{grad}"] & \underset{\text{exact}}{C^\infty(U,\mathbb{R}^3)} \arrow[u] \arrow[r, "\mathrm{curl}"] & \underset{\text{exact}}{C^\infty(U,\mathbb{R}^3)} \arrow[u] \arrow[r, "\mathrm{div}"] & C^\infty(U,\mathbb{R}) \arrow[u]
\end{tikzcd}$$

Figure 2: The updated commutative diagram for $U$ star-shaped, now with divergence.

*Proof.* Since $U$ is star-shaped by assumption (and therefore connected), we immediately have $H^0(U)=\mathbb{R}$ and $H^1(U)=0$ by our previous theorems. We want to show that $H^2(U)=0$, or $\ker(\operatorname{div})=\operatorname{im}(\operatorname{curl})$. Since $\operatorname{div}(\operatorname{im}(\operatorname{curl}))=0$, we have $\operatorname{im}(\operatorname{curl})\subseteq\ker(\operatorname{div})$. So the goal has been reduce to showing that $\ker(\operatorname{div})\subseteq\operatorname{im}(\operatorname{curl})$. To accomplish this, it suffices to exhibit a smooth function $U\subseteq\mathbb{R}^3\to\mathbb{R}^3$ such that the curl of this function is equal to some chosen element of $\ker(\operatorname{div})$.

Assume $U$ is star-shaped with respect to 0, and let $F\colon U\to\mathbb{R}^3$ such that $\operatorname{div} F=0$. Consider $G\colon U\to\mathbb{R}^3$ defined by

$$G(\mathbf{x})=\int_0^1 (F(t\mathbf{x})\times t\mathbf{x})\,dt.$$

Then if $\mathbf{x}=(x_1,x_2,x_3)$, $F=(f_1,f_2,f_3)$, we have

$$\begin{aligned}
\operatorname{curl}(F(t\mathbf{x})\times t\mathbf{x}) &= \operatorname{curl}((f_1(tx_1),f_2(tx_2),f_3(tx_3))\times(tx_1,tx_2,tx_3)) \\
&= \operatorname{curl}(f_2(tx_2)tx_3-f_3(tx_3)tx_2, f_3(tx_3)tx_1-f_1(tx_1)tx_3, f_1(tx_1)tx_2-f_2(tx_2)tx_1) \\
&= \left(\left(tf_1(tx_1)-tx_1\frac{\partial f_2}{\partial x_2}(tx_2)\right)-\left(tx_1\frac{\partial f_3}{\partial x_3}(tx_3)-tf_1(tx_1)\right),\cdots\right) \\
&= \left(2tf_1(tx_1)-tx_1\left(\frac{\partial f_2}{\partial x_2}(tx_1)+\frac{\partial f_3}{\partial x_3}(tx_3)\right),\cdots\right) \\
&= 2tF(t\mathbf{x})+{??} \\
&= \frac{d}{dt}(t^2F(t\mathbf{x})).
\end{aligned}$$

Therefore

$$\operatorname{curl} G(\mathbf{x})=\int_0^1 \frac{d}{dt}(t^2F(t\mathbf{x}))\,dt=F(\mathbf{x}).$$ ⊠

**Example 1.2.** If $U$ is not star-shaped then we can have nontrivial first and second cohomology groups. Consider $f\colon(\mathbb{R}^3\setminus S^1)\to\mathbb{R}^3$ by

$$f(x_1,x_2,x_3)=\left(\frac{-2x_1x_3}{x_3^2+(x_1^2+x_2^2-1)^2},\frac{-2x_2x_3}{x_3^2+(x_1^2+x_2^2-1)^2},\frac{x_1^2+x_2^2-1}{x_3^2+(x_1^2+x_2^2-1)^2}\right).$$

By some calculation we have $\operatorname{curl}(f)=0$. So $f$ defines some element $[f]\in H^1(U)$. To show $[f]$ is nontrivial, we integrate along a curve $\gamma\subseteq U$ "linked" to the missing $S^1$. Define $\gamma(t)=\left(\sqrt{1+\cos t},0,\sin t\right)$ for $t\in[-\pi,\pi]$. Assume $\operatorname{grad}(F)=f$ for some function $F$. One one hand, we have

$$\int_{-\pi+\varepsilon}^{\pi-\varepsilon}\frac{d}{dt}F(\gamma(t))\,dt=F(\gamma(\pi-\varepsilon))-F(\gamma(-\pi+\varepsilon))\to 0 \quad \text{for } \varepsilon\to 0,$$

and on the other hand we have by the chain rule

$$\frac{d}{dt}F(\gamma(t)) = f_1(\gamma(t))\cdot\gamma_1'(t)+\cdots = \sin^2 t + 0 + \cos^2 t = 1.$$

So the integral also converges to $2\pi$, a contradiction.

**Example 1.3.** Let $U \subseteq \mathbb{R}^k$ be open and $X : U \to \mathbb{R}^k$ be smooth ($X$ is a smooth vector field). The **energy** $A_\gamma(X)$ of $X$ along a smooth curve $\gamma : [a,b] \to U$ is defined by

$$A_\gamma(X) = \int_a^b \langle X\circ\gamma(t), \gamma'(t)\rangle\, dt.$$

If $X = \operatorname{grad}(\Phi)$ and $\Phi_{\gamma(a)} = \Phi_{\gamma(b)}$, then the energy of $X$ is zero, since

$$\langle X\circ\gamma(t), \gamma'(t)\rangle = \frac{d}{dt}\Phi(\gamma(t)).$$

Lecture 2

# Some algebra

## 2.1 The alternating algebra

Let $V$ be a real vector space. A map $f : \overbrace{V\times V\times\cdots\times V}^{k\text{ times}} \to \mathbb{R}$ is **k-linear** (or multilinear) if $f$ is linear in each factor.

**Definition 2.1.** A $k$-linear map $\omega : V^k \to \mathbb{R}$ is **alternating** if $\omega(\xi_1,\cdots,\xi_k) = 0$ whenever $\xi_i = \xi_j$ for some pair $i \neq j$. Denote the vector space of alternating $k$-linear maps as $A_k(V)$.

Note that $A_k(V) = 0$ if $k > \dim V$, since two vectors in the domain have to be linearly dependent. Recall that $\operatorname{sgn} : S_k \to \{\pm 1\}$ is a homomorphism, since $\operatorname{sgn}(\sigma\circ\tau) = \operatorname{sgn}(\sigma)\circ\operatorname{sgn}(\tau)$.

**Lemma 2.1.** *If $\omega \in A_k(V)$ and $\sigma \in S_k$, then*

$$\omega(\xi_{\sigma(1)},\cdots,\xi_{\sigma(k)}) = \operatorname{sgn}(\sigma)\omega(\xi_1,\cdots,\xi_k).$$

*Proof.* It is sufficient to show this is true for $\sigma = (i,j)$. Let $\omega_{i,j}(\xi,\xi') = \omega(\xi_1,\cdots,\xi,\cdots,\xi',\cdots,\xi_k)$, where $\xi$ and $\xi'$ occur at positions $i,j$ respectively. The remaining $\xi_\nu \in V$ are arbitrary fixed vectors. Now $\omega_{i,j} \in A_2(V)$ since $\omega \in A_k(V)$, so $\omega_{i,j}(\xi_i+\xi_j, \xi_i+\xi_j) = 0$. By bilinearity, we have $\omega_{i,j}(\xi_i,\xi_j) + \omega_{i,j}(\xi_j,\xi_i) = 0$, and so $\omega_{i,j}(\xi_i,\xi_j) = -\omega_{i,j}(\xi_j,\xi_i) = \operatorname{sgn}(\sigma)\omega_{i,j}(\xi_j,\xi_i)$. ⊠

**Example 2.1.** If $V = \mathbb{R}^k$ and $\xi_i = (\xi_{i1},\cdots,\xi_{ik})$, the determinant function $(\xi_1,\cdots,\xi_k) \mapsto \det(\xi_{ij})$ is alternating.

**Definition 2.2.** A **(p, q)-shuffle** $\sigma$ is a permutation in $S_{p+q}$ such that $\sigma(1) < \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(p+q)$. Denote the set of all $(p,q)$-shuffles by $S_{(p,q)}$. Since a $(p,q)$-shuffle is uniquely determined by the set $\{\sigma(1),\cdots\sigma(p)\}$, to form $S_{(p,q)}$ we choose subsets of order $p$ from $S_{p+q}$. So $|S_{(p,q)}| = \binom{p+q}{p}$.

**Definition 2.3.** For $\omega_1 \in A_p(V)$ and $\omega_2 \in A_q(V)$, define

$$(\omega_1\wedge\omega_2)(\xi_1,\cdots,\xi_{p+q}) = \sum_{\sigma\in S_{(p,q)}} \operatorname{sgn}(\sigma)\omega_1\left(\xi_{\sigma(1)},\cdots,\xi_{\sigma(p)}\right)\omega_2\left(\xi_{\sigma(p+1)},\cdots,\xi_{\sigma(p+q)}\right).$$

Note that $\omega_1\wedge\omega_2$ is $(p+q)$-linear. This product is called the **exterior product** or **wedge product**.

**Remark 2.1.** Often you also see the exterior product defined as

$$\omega_1 \wedge \omega_2(\xi_1, \cdots, \xi_{p+q}) := \frac{1}{p!q!} \sum_{\sigma \in S_{p+q}} \operatorname{sgn}(\sigma)\omega_1(\xi_{\sigma(1)}, \cdots, \xi_{\sigma(p)})\omega_2(\xi_{\sigma(p+1)}, \cdots, \xi_{\sigma(p+q)}).$$

This definition compensates for the $|S_p| = p!$ and $|S_q| = q!$ repetitions by dividing by them in the coefficient.

**Lemma 2.2.** *If $\omega_1 \in A_p(V)$ and $\omega_2 \in A_q(V)$, then $\omega_1 \wedge \omega_2 \in A_{p+q}(V)$.*

*Proof.* We show that $(\omega_1 \wedge \omega_2)(\xi_1, \xi_2, \cdots, \xi_{p+q}) = 0$ when $\xi_1 = \xi_2$. Let $S_{12} = \{\sigma \in S_{(p,q)} \mid \sigma(1) = 1, \sigma(p+1) = 2\}$, $S_{21} = \{\sigma \in S_{(p,q)} \mid \sigma(1) = 2, \sigma(p+1) = 1\}$, and $S_0 = S_{(p,q)}$ todo:algebra proof ☒

**Lemma 2.3.** *A $k$-linear map $\omega$ is alternating if $\omega(\xi_1, \cdots, \xi_k) = 0$ for all $k$-tuples with $\xi_i = \xi_{i+1}$ for some $1 \le i \le k-1$.*

*Proof.* Recall that $S_k$ is generated by the transpositions $(i, i+1)$, and so by Lemma 2.1, we have e

$$\omega(\xi_1, \cdots, \xi_i, \xi_{i+1}, \cdots, \xi_k) = -\omega(\xi_1, \cdots, \xi_{i+1}, \xi_i, \cdots, \xi_k).$$

Then Lemma 2.1 holds for all $\sigma \in S_k$, so $\omega$ is alternating.[2] ☒

**Lemma 2.4.** *The exterior product is anticommutative. That is, for $\omega_1 \in A_P(V)$ and $\omega_2 \in A_q(V)$, we have $\omega_1 \wedge \omega_2 = (-1)^{pq}\omega_2 \wedge \omega_1$.*

*Proof.* Define $\tau \in S_{p+q}$ to be the permutation

$$\tau = \begin{pmatrix} 1 & \cdots & q & q+1 & \cdots & q+p \\ p+1 & \cdots & p+q & 1 & \cdots & p \end{pmatrix}.$$

For any $\xi_1, \cdots, \xi_{p+q} \in V$,

$$\begin{aligned} \omega_1 \wedge \omega_2(\xi_1, \cdots, \xi_{p+q}) &= \sum_{\sigma \in S_{p+q}} (\operatorname{sgn} \sigma) f(\xi_{\sigma(1)}, \cdots, \xi_{\sigma(p)}) g(\xi_{\sigma(p+1)}, \cdots, \xi_{p\sigma(p+q)}) \\ &= \sum_{\sigma \in S_{p+q}} (\operatorname{sgn} \sigma) f(\xi_{\sigma\tau(q+1)}, \cdots, \xi_{\sigma\tau(q+p)}) g(\xi_{\sigma\tau(1)}, \cdots, \xi_{\sigma\tau(q)}) \\ &= (\operatorname{sgn} \tau) \sum_{\sigma \in S_{p+q}} (\operatorname{sgn} \sigma\tau) g(\xi_{\sigma\tau(1)}, \cdots, \xi_{\sigma\tau(q)}) f(\xi_{\sigma\tau(q+1)}, \cdots, \xi_{\sigma\tau(q+p)}) \\ &= (\operatorname{sgn} \tau) A(g \otimes f)(\xi_1, \cdots, \xi_{p+q}). \end{aligned}$$

todo:adapt this proof to the $(p, q)$-shuffle definition ☒

**Lemma 2.5.** *The exterior product is associative. That is, for $\omega_1 \in A_p(V)$, $\omega_2 \in A_q(V)$, $\omega_3 \in A_r(V)$, we have*

$$\omega_1 \wedge (\omega_2 \wedge \omega_3) = (\omega_1 \wedge \omega_2) \wedge \omega_3.$$

*Proof.* todo:spending less time on the algebra to get to the good stuff ☒

On top of making sure $A_k(V)$ is closed under multiplication and being associative, the exterior product is also associative and satisfies homogeneity, making it $A_k(V)$ into an algebra. What's an algebra? An $\mathbb{R}$-**algebra** $A$ is a real vector space with an associative bilinear map $\mu: A \times A \to A$. The algebra is **unitary** if there exists a unit element (say 1) such that $\mu(1, a) = \mu(a, 1) = a$ for all $a \in A$.

**Definition 2.4.**

(i) A **graded** $\mathbb{R}$-**algebra** $A_*$ is a sequence of vector spaces $A_k$, $k = 0, 1, \cdots$ and bilinear maps $\mu: A_k \times A_\ell \to A_{k+\ell}$ which are associative.

(ii) The graded algebra $A_*$ is **connected** if there exists a unit element $1 \in A_0$, and the map $\varepsilon: \mathbb{R} \to A_0$, $r \mapsto r \cdot 1$ is an isomorphism.

[2] Isn't this lemma true by definition?

(iii) The graded algebra $A_*$ is **commutative** (resp **anti-commutative**) if $\mu(a,b) = (-1)^{k\ell}\mu(b,a)$ for $a \in A_k$ and $b \in A_\ell$.

Elements in $A_k$ are said to have degree $k$.

Note that $A_k(V)$ is a real vector space since

$$(\omega_1+\omega_2)(\xi_1,\cdots,\xi_k) = \omega_1(\xi_1,\cdots,\xi_k)+\omega_2(\xi_1,\cdots,\xi_k),$$
$$(\lambda\omega) = (\xi_1,\cdots,\xi_k) = \lambda\omega(\xi_1,\cdots\xi_k), \quad \lambda \in \mathbb{R}.$$

**Theorem 2.1.** *$A_*(V)$ with the exterior product is an anti-commutative and connected graded algebra.*

*Proof.* Set $A_0(V) = \mathbb{R}$, since maps that take no vectors and output a scalar are just scalars themselves. Expand the product to $A_0(V) \times A_p(V)$ using the vector space structure. We have seen above that the exterior product is closed, associative, distributive, and anticommutative. ⊠

$A_*(V)$ is the **exterior algebra** or **alternating algebra** associated with $V$. Elements of $A_1(V)$ are called **1-forms**.

**Lemma 2.6.** *For $1-$forms $\omega_1,\cdots,\omega_p \in A_1(V)$, we have*

$$(\omega_1\wedge\cdots\wedge\omega_p)(\xi_1,\cdots,\xi_p) = \det\begin{pmatrix} \omega_1(\xi_1) & \omega_1(\xi_2) & \cdots & \omega_1(\xi_p) \\ \omega_2(\xi_1) & \omega_2(\xi_2) & \cdots & \omega_2(\xi_p) \\ \vdots & \vdots & & \vdots \\ \omega_p(\xi_1) & \omega_p(\xi_2) & \cdots & \omega_p(\xi_p) \end{pmatrix}.$$

*Proof.* We use induction on $p$. If $p=2$, then the two elements $(12),(21)$ of $S_2$ are $(1,1)$-shuffles. So $(\omega_1 \wedge \omega_2)(\xi_1,\xi_2) = \omega_1(\xi_1)\omega_2(\xi_2) - \omega_1(\xi_2)\omega_2(\xi_1) = \det\left(\begin{smallmatrix} \omega_1(\xi_1) & \omega_1(\xi_2) \\ \omega_2(\xi_1) & \omega_2(\xi_2)\end{smallmatrix}\right)$. Now

$$\omega_1\wedge(\omega_2\wedge\cdots\wedge\omega_p)(\xi_1,\cdots,\xi_p) = \sum_{j=1}^{p}(-1)\omega_1(\xi_j)(\omega_2\wedge\cdots\wedge\omega_p)\big(\xi_1,\cdots,\hat{\xi}_j,\cdots,\xi_p\big).$$

Expanding the determinant along the first row gives our result. ⊠

This lemma shows that if the 1-forms $\omega_1,\cdots,\omega_p$ are linearly independent, then $\omega_1\wedge\cdots\wedge\omega_p \neq 0$. This is an equivalence: we can choose elements $\xi_i \in V$ with $\omega_i(\xi_j) = 0$ for $i \neq j$ and $\omega_j(\xi_j) = 1$, which implies that $\det(\omega_i(\xi_j)) = 1$. Conversely, if the $\omega_i$ were linearly dependent, we could write $\omega_p = \sum_{i=1}^{p-1} r_i\omega_i$. So the determinant in the previous lemma would have two equal rows and be zero. To summarize:

**Lemma 2.7.** *For 1-forms $\omega_1,\cdots,\omega_p$ on $V$, we have $\omega_1\wedge\cdots\wedge\omega_p \neq 0$ iff they are linearly independent.*

**Theorem 2.2.** *For $\{e_i\}$ a basis of $V$ and $\{\phi_i\}$ the dual basis of $A_1(V)$ (as i varies over n), we have*

$$\{\phi_{\sigma_1}\wedge\phi_{\sigma(2)}\wedge\cdots\wedge\phi_{\sigma(p)}\}_{\sigma\in S_{(p,n-p)}}$$

*a basis of $A_p(V)$. In particular, $\dim A_p(V) = \binom{\dim V}{p}$.*

*Proof.* todo:less time on algebra ⊠

This tells us that $A_n(V) \cong \mathbb{R}$ if $n = \dim V$ (since they're both one dimensional real vector spaces, $\binom{n}{n} = 1$) and $A_p(V) = 0$ for $p > n$ (since two factors will be the same). A linear map $f: V \to W$ induces the linear map

$$A_p(f)\colon A_p(W) \to A_p(V)$$

by setting $A_p(f)(\omega(\xi_1,\cdots,\xi_p)) = \omega(f(\xi_1),\cdots,f(\xi_p))$. We have $A_p(g\circ f) = A_p(f)\circ A_p(g)$, and $A_p(\mathrm{id}) = \mathrm{id}$. This is equivalent to saying that $A_p(-)$ is a **contravariant functor**. For $\dim V = n$, $f: V \to V$ linear, the induced map $A_n(f)\colon A_n(V) \to A_n(V)$ is a linear endormorphism of a 1-dimensional vector space, and is therefore just scalar multiplication. It follows from the theorem below that this scalar is $\det f$.

**Theorem 2.3.** *The characteristic polynomial of a linear endomorphism $f: V \to V$ is given by*

$$\det(f-t) = \sum_{i=0}^{n}(-1)^i\,\mathrm{tr}\,(A_{n-i}(f))\,t^i.$$

*Proof.* todo:algebra ⊠

## 2.2 The exterior derivative

Let $U$ denote an open set in $\mathbb{R}^n$, $\{e_1, \cdots, e_n\}$ the standard basis and $\{\phi_1, \cdots, \phi_n\}$ the dual basis of $A_1(\mathbb{R}^n)$ (or the basis for the dual space to $\mathbb{R}^n$).

**Definition 2.5.** A **differential p-form** on $U$ is a smooth map $\omega \colon U \to A_p(\mathbb{R}^n)$. The vector space of all such maps is denoted by $\Omega^p(U)$.

If $p = 0$, then $A_0(\mathbb{R}^n) = \mathbb{R}$, and $\Omega^0(U)$ is just the set of smooth functions on $U, C^\infty(U, \mathbb{R})$. The derivative of a smooth map $\omega \colon U \to A_p(\mathbb{R}^n)$ is denoted $D\omega$, and is the linear map

$$D_x\omega \colon \mathbb{R}^n \to A_p(\mathbb{R}^n), \quad e_i \mapsto \frac{d}{dt}\omega(x + te_i)_{t=0} = \frac{\partial \omega}{\partial x_i}(x).$$

Let $I = (i_1, \cdots, i_p)$, and write $\phi_I$[3] for $\phi_{i_1} \wedge \cdots \wedge \phi_{i_p}$. Then we have the basis $\phi_I$ for $A_p(\mathbb{R}^n)$ as $I$ runs over all sequences of length $p \leq n$. So every $\omega \in \Omega^p(U)$ can be written in the form $\omega(w) = \sum \omega_I(x)\phi_I$, where the $\omega_I$ are smooth real-valued functions of $x \in U$. The differential $D_x\omega$ is the linear map

$$D_x\omega(e_j) = \sum_I \frac{\partial \omega_I}{\partial x_j}(x)\phi_I, \quad j = 1, \cdots, n.$$

The function $x \mapsto D_x\omega$ is a smooth map from $U$ to $\mathrm{Hom}(\mathbb{R}^n, A_p(\mathbb{R}^n))$. todo:why? how exactly? difference between thsi and derivative?

**Definition 2.6.** The **exterior differential** $d \colon \Omega^p(U) \to \Omega^{p+1}(U)$ is the linear operator

$$d_x\omega(\xi_1, \cdots, \xi_{p+1}) = \sum_{\ell=1}^{p+1} (-1)^{\ell-1} D_x\omega(\xi_\ell)(\xi_1, \cdots, \hat{\xi}_\ell, \cdots, \xi_{p+1})$$

where $(\xi_1, \cdots, \hat{\xi}_\ell, \cdots, \xi_{p+1}) = (\xi_1, \cdots, \xi_{\ell-1}, \xi_{\ell+1}, \cdots, \xi_{p+1})$. todo:what?

The result lies in $\Omega^{p+1}(U)$ by Lemma 2.3. If $\xi_i = \xi_{i+1}$, then

$$\begin{aligned}
&\sum_{\ell=1}^{p+1} (-1)^{\ell-1} D_x\omega(\xi_\ell)(\xi_1, \cdots, \hat{\xi}_\ell, \cdots, \xi_{p+1}) \\
&=(-1)^{i-1} D_x\omega(\xi_i)(\xi_1, \cdots, \hat{\xi}_i, \cdots, \xi_{p+1}) \\
&\quad + (-1)^i D_x\omega(\xi_{i+1})(\xi_1, \cdots, \hat{\xi}_{i+1}, \cdots, \xi_{p+1}) \\
&=0.
\end{aligned}$$

In the second step, the rest of the terms cancel out by properties of the exterior product, since they all contain both $\xi_i$ and $\xi_{i+1}$. The final term also cancels out since $(\xi_1, \cdots, \hat{\xi}_i, \cdots, \xi_{p+1}) = (\xi_1, \cdots, \hat{\xi}_{i+1}, \cdots, \xi_{p+1})$.

**Example 2.2.** Let $x_i \colon U \to \mathbb{R}$ be $i$th projection. Then $dx_1 \in \Omega^1(U)$ is the constant map $dx_i \colon x \to \phi_i$, which follows from the definition of the differential. In general, for $f \in \Omega^0(U)$, we have

$$d_x f(\zeta) = \frac{\partial f}{\partial x_1}(x)\zeta^1 + \cdots + \frac{\partial f}{\partial x_n}(x)\zeta^n.$$

**Lemma 2.8.** *If $\omega(x) = f(x)\phi_I$, then $d_x\omega = d_x f \wedge \phi_I$.*

*Proof.* Note that

$$D_x\omega(\zeta) = (D_x f)(\zeta)\phi_I = \left(\frac{\partial f}{\partial x_1}\zeta^1 + \cdots + \frac{\partial f}{\partial x_n}\zeta^n\right)\phi_I = d_x f(\zeta)\phi_I.$$

[3] It slightly annoys me that indices aren't in the right place, but I don't want to make any mistakes deviating too far from the book, so they stay at the bottom for covectors.

So by the definition of the exterior derivative, we have

$$
\begin{aligned}
d_x\omega(\xi_1,\cdots,\xi_{p+1}) &= \sum_{k=1}^{p+1}(-1)^{k-1}d_xf(\xi_k)\phi_I(\xi_1,\cdots,\hat{\xi}_k,\cdots,\xi_{p+1})\\
&= [d_xf\wedge\phi_I](\xi_1,\cdots,\xi_{p+1}).
\end{aligned}
$$

⊠

todo:this entire proof? For $\phi_I\in A_p(\mathbb{R}^n)$, we have $\phi_k\wedge\phi_I=0$ if $k\in I$, and $(-1)^r\phi_J$ if $k\notin I$, where $r$ is determined by $i_r<k<i_{r+1}$ and $J=(i_1,\cdots,i_r,k,\cdots,i_p)$.

**Lemma 2.9.** *For $p\geq 0$, the composition $\Omega^p(U)\to\Omega^{p+1}(U)\to\Omega^{p+2}(U)$ is identically zero.*

*Proof.* Let $\omega=f\phi_I$. Then $d\omega=df\wedge\phi_I=\frac{\partial f}{\partial x_1}\phi_1\wedge\phi_I+\cdots+\frac{\partial f}{\partial x_n}\phi_n\wedge\phi_I$. todo:alternating terms?? does $I$ denote one sequence or several? Since $\phi_i\wedge\phi_i=0$ and $\phi_i\wedge\phi_j=-\phi_j\wedge\phi_i$, we have

$$
\begin{aligned}
d^2\omega &= \sum_{i,j=1}^{n}\frac{\partial^2 f}{\partial x_i\partial x_j}\phi_i\wedge(\phi_j\wedge\phi_I)\\
&= \sum_{i<j}\left(\frac{\partial^2 f}{\partial x_i\partial x_j}-\frac{\partial^2 f}{\partial x_j\partial x_j}\right)\phi_i\wedge\phi_j\wedge\phi_I=0.
\end{aligned}
$$

⊠

The exterior product on $A_*(\mathbb{R}^n)$ induces an exterior product on $\Omega^*(U)$ by defining $(\omega_1\wedge\omega_2)(x)=\omega_1(x)\wedge\omega_2(x)$. The exterior product of a $p$-form and $q$-form is a $(p+q)$-form, so it induces a bilinear map $\wedge\colon\Omega^p(U)\times\Omega^q(U)\to\Omega^{p+q}(U)$. Then for $f\in C^\infty(U,\mathbb{R})$, we have $(f\omega_1)\wedge\omega_2=f(\omega_1\wedge\omega_2)=\omega_1\wedge f\omega_2$. Note that $f\wedge\omega=f\omega$ when $f\in\Omega^0(U)$ and $\omega\in\Omega^p(U)$.

**Lemma 2.10.** *For $\omega_1\in\Omega^p(U)$ and $\omega_2\in\Omega^q(U)$,*

$$
d(\omega_1\wedge\omega_2)=d\omega_1\wedge\omega_2+(-1)^p\omega_1\wedge d\omega_2.
$$

*Proof.* It suffices to show this holds for $\omega_1=f\phi_I$ and $\omega_2=g\phi_J$. Since $\omega_1\wedge\omega_2=fg\phi_I\wedge\phi_J$, we have

$$
\begin{aligned}
d(\omega_1\wedge\omega_2) &= d(fg)\wedge\phi_I\wedge\phi_J=((df)g+f\,dg)\wedge\phi_I\wedge\phi_J\\
&= df\,g\wedge\phi_I\wedge\phi_J+f\,dg\wedge\phi_I\wedge\phi_J\\
&= df\wedge\phi_I\wedge g\phi_J+(-1)^p f\phi_I\wedge dg\wedge\phi_J\\
&= d\omega_1\wedge\omega_2+(-1)^p\omega_1\wedge d\omega_2.
\end{aligned}
$$

⊠

## 2.3 Finally, de Rham cohomology

In short, we have a new anti-commutative algebra $\Omega^*(U)$ with a *differential* (or boundary) operator

$$
d\colon\Omega^*(U)\to\Omega^{*+1}(U),\quad d\circ d=0,
$$

and $d$ is a *derivation* (since $d(\omega_1\wedge\omega_2)=d\omega_1\wedge\omega_2+d\omega_2\wedge\omega_1$ by Lemma 2.10 and anticommutativity). Then $(\Omega^*(U),d)$ is an *commutative differential graded algebra*[4], called the **de Rham complex** of $U$.

**Theorem 2.4.** *There is precisely one linear operator $d\colon\Omega^p\to\Omega^{p+1}(U)$, $p=0,1,\cdots$, such that*

*(i)* $f\in\Omega^0(U)$, $df=\frac{\partial f}{\partial x_1}\phi_1+\cdots+\frac{\partial f}{\partial x_n}\phi_n$,
*(ii)* $d\circ d=0$,
*(iii)* $d(\omega_1\wedge\omega_2)=d\omega_1\wedge\omega_2+(-1)^p\omega_1\wedge d\omega_2$ *if* $\omega_1\in\Omega^p(U)$.

*Proof.* We know that the exterior differential $d$ satisfies these properties. To show uniqueness, say $d'$ satisfies these properties: we will show it has to be the exterior derivative. (i) tells us that $d=d'$ on $\Omega^0(U)$ (since it characterizes smooth functions on $U$), in particular $d'x_i=dx_i=\phi_i$. Since $d'\circ d'=0$, then $d'\phi_i=d'(d'(x_i))=0$, and $d'\phi_I=0$. Let $\omega=f\phi_I=f\wedge\phi_I$ for $f\in C^\infty(U,\mathbb{R})$. Then

$$
d'\omega=d'f\wedge\phi_I+f\wedge d'\phi_I=d'f\wedge\phi_I=df\wedge\phi_I=d\omega.
$$

⊠

[4] Strangely, even though the algebra is anticommutative, we call it commutative graded. This is just convention.

**Example 2.3.** Let us show a concrete example. Let $U \subseteq \mathbb{R}^3$, then $d : \Omega^1(U) \to \Omega^2(U)$ looks like the following:

$$d(f_1\phi_1 + f_2\phi_2 + f_3\phi_3) = df_1 \wedge \phi_1 + df_2 \wedge \phi_2 + df_3 \wedge \phi_3 =$$
$$\left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right)\phi_1 \wedge \phi_2 + \left(\frac{\partial f_3}{\partial x_3} - \frac{\partial f_2}{\partial x_3}\right)\phi_2 \wedge \phi_3 + \left(\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}\right)\phi_3 \wedge \phi_1.$$

todo:at this point, the hard-to-read book was abandoned for bott and tu

Lecture 3

# De Rham Cohomology

Switching gears, the reference book has changed to Bott and Tu.

## 3.1 The de Rham Complex on $\mathbb{R}^n$

If $x_1, \cdots, x_n$ are the standard coordinates on $\mathbb{R}^n$, define $\Omega^*$ to be the algebra over $\mathbb{R}$ generated by $dx_1, \cdots, dx_n$ with the relations

$$\begin{cases} (dx_i)^2 = 0, \\ dx_i dx_j = -dx_j dx_i,\ i \neq j. \end{cases}$$

As a real vector space this has basis $1, dx_i, dx_i dx_j, dx_i dx_j dx_k, \cdots, dx_1 \cdots dx_n$, where $i < j, i < j < k$ (or the $(*, n)$-shuffles). The $C^\infty$ **differential forms** on $\mathbb{R}^n$ are elements of $\Omega^*(\mathbb{R}^n) = \{C^\infty \text{ functions on } \mathbb{R}^n\} \bigotimes_{\mathbb{R}} \Omega^*$. Recall that the tensor product of two $R$-algebras $A, B$ has basis $a_i \otimes b_j$, where multiplication is defined by $(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 b_1 \otimes a_2 b_2$. So a form $\omega$ can be uniquely written as $\sum f_{i_1 \cdots i_q} dx_{i_1} \cdots dx_{i_q}$, where the coefficients $f_{i_1 \cdots i_q}$ are smooth functions. The multi-index notation simplifies this to $\omega = \sum f_I dx_I$. The algebra $\Omega^*(\mathbb{R}^n) = \bigoplus_{q=0}^n \Omega^q(\mathbb{R}^n)$ is naturally graded, where $\Omega^q$ is the space of $C^\infty$ $q$-forms on $\mathbb{R}^n$. There is a *differential operator*

$$d : \Omega^q(\mathbb{R}^n) \to \Omega^{q+1}(\mathbb{R}^n)$$

defined as follows:

(i) if $f \in \Omega^0(\mathbb{R}^n)$, then $df = \sum \frac{\partial f}{\partial x^i} dx_i$,
(ii) if $\omega = \sum f_I dx_I$, then $d\omega = \sum df_I dx_I$.

We call this differential operator **exterior differention**.

**Example 3.1.** If $\Omega = x\,dy$, then $d\omega = dx\,dy$. On $\mathbb{R}^3$, $\Omega^0(\mathbb{R}^3)$ and $\Omega^3(\mathbb{R}^3)$ are both 1-dimensional and $\Omega^1(\mathbb{R}^3)$ and $\Omega^2(\mathbb{R}^3)$ are each 3-dimensional over the $C^\infty$ functions, so we identify

$$\begin{array}{ccccc} \underset{f}{\{\text{functions}\}} & \simeq & \underset{f}{\{\text{0-forms}\}} & \simeq & \underset{f\,dx\,dy\,dz}{\{\text{3-forms }\}}, \\ \underset{X=(f_1,f_2,f_3)}{\{\text{vector fields}\}} & \simeq & \underset{f_1dx+f_2dy+f_3dz}{\{\text{1-forms}\}} & \simeq & \underset{f_1dy\,dz-f_2dx\,dz+f_3dx\,dy}{\{\text{2-forms }\}} \end{array}.$$

So for functions,

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz,$$

for 1-forms

$$d(f_1dx + f_2dy + f_3dz) = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)dy\,dz - \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)dx\,dz + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)dx\,dy,$$

and for 2-forms

$$d(f_1\,dy\,dz - f_2\,dx\,dz + f_3\,dx\,dy = \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right)dx\,dy\,dz.$$

So $d(\text{0-forms}) = \text{gradient}$, $d(\text{1-forms}) = \text{curl}$, and $d(\text{2-forms}) = \text{divergence}$.todo:yay worked it out! transcribe

**Definition 3.1.** Define the **wedge product** of two differential forms, written $\tau \wedge \omega$ for $\tau = \sum f_I dx_I$, $\omega = \sum g_J dx_J$ by

$$\tau \wedge \omega = \sum f_I g_J \, dx_I \, dx_J.$$

Note that $\tau \wedge \omega = (-1)^{\deg \tau \deg \omega} \omega \wedge \tau$.

**Proposition 3.1.** *$d$ is an antiderivation, i.e.,*

$$d(\tau \wedge \omega) = (d\tau) \wedge \omega + (-1)^{\deg \tau} \tau \wedge d\omega.$$

*Proof.* By linearity, we check on just the monomials $\tau = f_I \, dx_I$, $\omega = g_J \, dx_J$. Then

$$d(\tau \wedge \omega) = d(f_1 g_J) dx_I \, dx_J = (df_I) g_J \, dx_I \, dx_J + f_I \, dg_J \, dx_I \, dx_J = (d\tau) \wedge \omega + (-1)^{\deg \tau} \tau \wedge d\omega. \qquad \boxtimes$$

**Proposition 3.2.** $d^2 = 0$.

*Proof.* On functions,

$$d^2 f = d\left(\sum_i \frac{\partial f}{\partial x_i} dx_i\right) = \sum_{i,j} \frac{\partial^2 f}{\partial x_j \partial x_i} dx_j dx_i = 0,$$

since the factors $\partial^2/\partial x_j \partial x_i$ are symmetric in $i, j$ (mixed partials commute) while the $dx_j dx_i$ are skew-symmetric in $i, j$, hence $d^2 f = 0$. On forms $\omega = f_I \, dx_I$,

$$d^2 \omega = d^2(f_I \, dx_I) = d(df_I dx_I) = 0. \qquad \boxtimes$$

The complex $\Omega^*(\mathbb{R}^n)$ with the differential operator $d$ is the **de Rham complex** on $\mathbb{R}^n$. The kernel of $d$ are **closed forms** and the image of $d$ are **exact forms**. You can view the de Rham complex as a set of differential equations with solutions the closed forms. For example, finding a closed 1-form $f \, dx + g \, dy$ on $\mathbb{R}^2$ is just like solving the differential equation $\partial g/\partial x - \partial f/\partial y = 0$. todo:where did the $dx \, dy$ term go?

Exact forms are automatically closed, since composing with $d$ again gives zero. These are the "trivial" or "uninteresting" solutions: the de Rham cohomology measures the size of the space of "interesting" solutions.

**Definition 3.2.** The $q$-th **de Rham cohomology** of $\mathbb{R}^n$ is the vector space

$$H^q_{DR}(\mathbb{R}^n) = \{\text{closed } q\text{-forms}\}/\{\text{exact } q\text{-forms}\}.$$

We often write $H^q(\mathbb{R}^n)$ in place of $H^q_{DR}(\mathbb{R}^n)$. Denote the cohomology class of a form by $[\omega]$.

All the definitions work just as well with an open subset $U \subseteq \mathbb{R}^n$. For example, define $\Omega^*(U)$ to be the algebra $\{C^\infty \text{ functions on } U\} \bigotimes_{\mathbb{R}} \Omega^*$.[5] Then we may speak of the de Rham cohomology $H^*_{DR}(U)$ of $U$.

**Example 3.2.** If $n = 0$, then

$$H^q(\text{point}) = \begin{cases} \mathbb{R}, & \text{if } q = 0, \\ 0, & \text{if } q > 0. \end{cases}$$

Since $\ker d \subseteq \Omega^0(\mathbb{R}^1)$ consists of constant functions, $H^0(\mathbb{R}^1) = \mathbb{R}$ as we've seen before, On $\Omega^1(\mathbb{R}^1)$, $\ker d$ consists of all the 1-forms, since every 1-form $f \, dx$ gets send to $\frac{\partial f}{\partial x} dx \, dx = 0$. For $\omega = g(x) \, dx$ a 1-form, consider $f = \int_0^x g(u) \, du$, so $df = g(x) \, dx$. Therefore every 1-form on $\mathbb{R}^1$ is exact and $H^1(\mathbb{R}^1) = 0$. If $U$ is a disjoint union of $m$ intervals on $\mathbb{R}$, then $H^0(U) = \mathbb{R}^m$, and $H^1(U) = 0$. In general,

$$H^*(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{in dimension } 0, \\ 0 & \text{otherwise.} \end{cases}$$

This result is called the *Poincaré lemma*.

[5] To avoid confusion, when the notation $C^\infty$ functions on $\{\text{set}\}$ is used, the codomain of the functions is just the base field, in this case $\mathbb{R}$.

## 3.2 Recap of (co)homological algebra

A direct sum of vector spaces $C = \bigoplus_{q\in\mathbb{Z}} C^q$ is called a **differential complex** if there are homomorphisms

$$\cdots \to C^{q-1} \xrightarrow{d} C^q \xrightarrow{d} C^{q+1} \to \cdots$$

such that $d^2 = 0$. We say that $d$ is the **differential operator** of the complex $C$. Note that the de Rham complex is a differential complex, and the exterior derivative is a differential operator on $\Omega^q(U)$. The **cohomology** of $C$ is the direct sum of vector spaces $H(C) = \bigoplus_{q\in\mathbb{Z}} H^q(C)$, where $H^q(C) = (\ker d \cap C^q)/(\operatorname{im} d \cap C^q)$. A map $f : A \to B$ between two differential complexes is a **chain map** if it commutes with the differential operators of $A$ and $B$, that is, $f\, d_A = d_B f$.

A sequence of vector spaces

$$\cdots \longrightarrow V_{i-1} \xrightarrow{f_{i-1}} V_i \xrightarrow{f_i} V_{i+1} \longrightarrow \cdots$$

is **exact** if for all $i$, $\ker f_i = \operatorname{im} f_{i-1}$. An exact sequence of the form $0 \to A \to B \to C \to 0$ is a **short exact sequence**. Given a short exact sequence of differential complexes

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

where $f$ and $g$ are chain maps, there is a long exact sequence of cohomology groups

$$\cdots \to H^q(A) \xrightarrow{f^*} H^q(B) \xrightarrow{g^*} H^q(C) \xrightarrow{d^*} H^{q+1}(A) \to \cdots$$

where $f^*, g^*$ are the induced homomomorphisms on cohomology, and $d^*[c]$ for $c \in C^q$ is obtained as follows:

$$\begin{tikzcd}
 & {} & {} & {} & \\
0 \arrow[r] & A^{q+1} \arrow[u] \arrow[r, "f"] & B^{q+1} \arrow[u] \arrow[r, "g"] & C^{q+1} \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & A^q \arrow[u, "d"] \arrow[r, "f"] & B^q \arrow[u, "d"] \arrow[r, "g"] & C^q \arrow[u, "d"] \arrow[r] & 0 \\
 & {} \arrow[u] & {} \arrow[u] & {} \arrow[u] &
\end{tikzcd}$$

By the fact that $g$ is onto, there exists a $b \in B^q$ such that $g(b) = c$. Since $g(db) = d(gb) = dc = 0$ (implying that $db \in \ker g$), we have $db = f(a)$ for some $a \in A^{q+1}$ by exactness. It is easily checked that $a$ is closed. $d^*[c]$ is defined to be the cohomology class $[a]$ in $H^{q+1}(A)$.

## 3.3 Compact supports

Recall the **support** of a continuous function $f$ on a topological space $X$ is the closure of the set on which $f$ is nonzero, that is, $\operatorname{Supp} f = \overline{\{p \in X \mid f(p) \neq 0\}}$. If when defining the de Rham complex we only consider the $C^\infty$ functions with compact support (equivalent to having bounded support in this case), the resulting complex is called the **de Rham complex $\Omega_c^*(\mathbb{R}^n)$ with compact supports**:

$$\Omega_c^*(\mathbb{R}^n) := \{C^\infty \text{ functions on } \mathbb{R}^n \text{ with compact support}\} \bigotimes_{\mathbb{R}} \Omega^*.$$

Denote the cohomology of this complex by $H_c^*(\mathbb{R}^n)$.

**Example 3.3.** First note that

$$H_c^*(\text{point}) = \begin{cases} \mathbb{R} & \text{in dimension } 0, \\ 0 & \text{elsewhere.} \end{cases}$$

The closed 0-forms on $\mathbb{R}^1$ are once again the constant functions, and since no constant function has compact support, $H_c^0(\mathbb{R}^1) = 0$ todo:more confusion about domains: doesn't $\mathbb{R} \to 0$ work? is $\emptyset$ compact? also, is compact support like the analogue of reduced homology? To compute $H_c^1(\mathbb{R}^1)$ consider the integration map

$$\int_{\mathbb{R}^1} : \Omega_c^1(\mathbb{R}^1) \longrightarrow \mathbb{R}^1.$$

todo:were we ever taught how to integrate on forms? This map is onto, and vanishes on the exact 1-forms $df$ where $f$ has compact support, for if the support of $f$ lies in the interior of $[a,b]$, then

$$\int_{\mathbb{R}^1} \frac{df}{dx} dx = \int_a^b \frac{df}{dx} dx = f(b) - f(a) = 0.$$

If $g(x)\,dx \in \Omega_c^1(\mathbb{R}^1)$ is also in $\ker \int_{\mathbb{R}^1}$, then $f(x) = \int_{-\infty}^x g(u)\,du$ will have compact support and $df = g(x)\,dx$. Hence $\ker \int_{\mathbb{R}^1}$ consists of precisely the exact forms and

$$H_c^1(\mathbb{R}^1) = \frac{\Omega_c^1(\mathbb{R}^1)}{\ker \int_{\mathbb{R}^1}} = \mathbb{R}^1.$$

To see this, todo:elaborate

**Remark 3.1.** If $g(x)\,dx \in \Omega_c^1(\mathbb{R}^1)$ does not have total integral 0, then $f(x) = \int_{-\infty}^x g(u)\,du$ will not have compact support and $g(x)\,dx$ will not be exact.

In general,

$$H_c^*(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{in dimension } n, \\ 0 & \text{otherwise.} \end{cases}$$

This is called the *Poincaré lemma for cohomology with compact support*.

**Example 3.4.** todo:comptue this with both regular (find the closed forms)+mayer vietoris, ask about summer drp To compute $H_{DR}^*(\mathbb{R}^2 \setminus \{p,q\})$ for $p,q \in \mathbb{R}^2$,

If $x_1, \cdots, x_m$ and $y_1, \cdots, y_n$ are the standard coordinates on $\mathbb{R}^m$ and $\mathbb{R}^n$, then a smooth map $f : \mathbb{R}^m \to \mathbb{R}^n$ incudes a pullback map on $C^\infty$ functions $f^* : \Omega^0(\mathbb{R}^n) \to \Omega^0(\mathbb{R}^m)$ defined by $f^*(g) = g \circ f$. To uniquely extend this pullback to all forms $f^* : \Omega^*(\mathbb{R}^n) \to \Omega^*(\mathbb{R}^m)$, consider

$$f^*\left(\sum g_I dy_{i_1} \cdots dy_{i_q}\right) = \sum (g_I \circ f) df_{i_1} \cdots df_{i_q},$$

where $f_i = y_i \circ f$ is the $i$th component of $f$.

**Proposition 3.3.** *$f^*$ commutes with $d$.*

*Proof.*

$$\begin{aligned} df^*(g_I dy_{i_1} \cdots dy_{i_q}) &= d((g_I \circ f) df_{i_1} \cdots df_{i_q}) = d(g_I \circ f) df_{i_1} \cdots df_{i_q}, \\ f^* d(g_I dy_{i_1} \cdots dy_{i_q}) &= f^*\left(\sum_{i=1}^n \frac{\partial g_I}{\partial y_i} dy_i dy_{i_1} \cdots dy_{i_q}\right) \\ &= \sum_{i=1}^n \left(\left(\frac{\partial g_I}{\partial y_i} \circ f\right) df_i\right) df_{i_1} \cdots df_{i_q} \\ &= d(g_I \circ f) df_{i_1} \cdots df_{i_q}. \end{aligned}$$

⊠

Lecture 4

# The Mayer-Vietoris Sequence

Note that $\Omega^*$ is a contravariant function from the category of Euclidian spaces $\{\mathbb{R}^n\}_{n\in\mathbb{Z}}$ and smooth maps $\mathbb{R}^m \to \mathbb{R}^n$ to the category of commutative differential graded algebras and their homomorphisms. We can extend the functor $\Omega^*$ to differentiable manifolds, where todo:stuff about pou's

## 4.1 Definition of the Mayer-Vietoris sequence

The Mayer-Vietoris sequence allows one to compute the cohomology of the union of two open sets. Suppose $M = U \cup V$, where $U, V$ are open. Then we have a sequence of inclusions

$$M \leftarrow U \amalg V \underset{\partial_1}{\overset{\partial_0}{\leftleftarrows}} U \cap V$$

where $\partial_0 : U \cap V \hookrightarrow V$, $\partial_1 : U \cap V \hookrightarrow U$ todo:how is this union disjoin? . Apply the contravariant functor $\Omega^*$ to get

$$\Omega^*(M) \to \Omega^*(U) \oplus \Omega^*(V) \underset{\partial_1^*}{\overset{\partial_0^*}{\rightrightarrows}} \Omega^*(U \cap V),$$

this splits into the **Mayer-Vietoris sequence**, given by

$$0 \longrightarrow \Omega^*(M) \longrightarrow \underset{(\omega,\tau)}{\Omega^*(U) \oplus \Omega^*(V)} \underset{\mapsto}{\longrightarrow} \underset{\tau-\omega}{\Omega^*(U \cap V)} \longrightarrow 0.$$

**Proposition 4.1.** *The Mayer-Vietoris sequence is exact.*