Differential Equations Notes

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Notes for Differential Equations (M 427J) with Dr. Tsishchanka at UT Austin. Only ever taken right before exams. Source code: https://git.simonxiang.xyz/math_notes/file/freshman_year/differential_equations/master_notes.tex.html.

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Lecture 1

First Order Linear Differential Equations (8/27/20)

1.1 Definitions

Definition 1.1 (Order)**.** We have the *order* of a differential equation the highest derivative of a function y that appears in the equation. For example, the order of the differential equation

$$\frac{dy}{dt} = 3y^2 \sin(t + y)$$

is 1, while the order of

$$\frac{d^3y}{dt^3} = e^{-y} + t + \frac{d^2y}{dt^2}$$

is 3. We would call the first example a *first-order differential equation* and the second a *third order differential equation*.

Definition 1.2 (Solution)**.** The *solution* of a differential equation is a continuous function $y(t)$ that together with its derivatives satisfies the given relationship.

Example 1.1. The function

$$y(t) = 2\sin t - \frac{1}{3}\cos 2t$$

is the solution of the second-order differential equation

$$\frac{d^2y}{dt^2} + y = \cos 2t$$

since

$$\begin{aligned} &\frac{d^2}{dt^2}\left(2\sin t - \frac{1}{3}\cos 2t\right) + \left(2\sin t - \frac{1}{3}\cos 2t\right) \\ &= \left(-2\sin t + \frac{4}{3}\cos 2t\right) + 2\sin t - \frac{1}{3}\cos 2t = \cos 2t. \end{aligned}$$

Goal: Given a differential equation of the form

$$\frac{dy}{dt} = f(t, y)$$

and the function $f(t, y)$, find all functions $y(t)$ that satisfy the equation above.

What we have: As of now, all we can solve is a differential equation of the form

$$\frac{dy}{dt} = g(t)$$

given $g(t)$ is integrable. Very sad!

Definition 1.3 (Linear ODE)**.** The general first-order linear differential equation is of the form

$$\frac{dy}{dt} + a(t)y = b(t),$$

where $a(t)$ and $b(t)$ are continuous (assumed to be functions of time).

1.2 Homogeneous Linear Ordinary Differential Equations

Definition 1.4 (Homogeneous Linear ODE)**.** The equation

$$\frac{dy}{dt} + a(t)y = 0$$

is called the *homogeneous* first-order linear differential equation, and the previous definition is called the *nonhomogeneous* first-order linear differential equation for $b(t)$ not necessarily zero.

Example 1.2. Let us solve the homogeneous first-order linear differential equation. Rewrite it in the form

$$\frac{\frac{dy}{dt}}{y} = -a(t).$$

Second, note that

$$\frac{\frac{dy}{dt}}{y} \equiv \frac{d}{dt} \ln|y(t)|.$$

Then we can write the differential equation in the form

$$\frac{d}{dt} \ln|y(t)| = -a(t),$$

so we have

$$\ln|y(t)| = -\int a(t)\,dt + c_1.$$

Continuing on,

$$|y(t)| = \exp\left(-\int a(t)\,dt + c_1\right) = c\exp\left(-\int a(t)\,dt\right)$$

or

$$\left| y(t)\exp\left(\int a(t)\,dt\right)\right| = c.$$

Now $y(t)\exp\left(\int a(t)\,dt\right)$ is continuous and we know its absolute value is constant which implies that the function itself is constant (which follows from the IVT, assuming $g(t_1) = c$ and $g(t_2) = -c$ for g a function, c a constant. So we have $y(t)\exp\left(\int a(t)\,dt\right) = c$, or

$$y(t) = c\exp\left(-\int a(t)\,dt\right). \tag{1}$$

Equation (1) is the *general solution* of the homogeneous equation. Note that there exist infinitely many solutions since for all c we have a distinct $y(t)$.

Example 1.3. To solve the Linear ODE

$$\frac{dy}{dt} + 2ty = 0,$$

simply apply Equation (1) to yield

$$y(t) = c\exp\left(-\int 2t\,dt\right) = c\exp\left(-t^2\right).$$

(This is taking too long! I'll type notes with less rigor next time).

1.3 Initial Value Problems

Usually scientists are not interested in the general solution given by Equation (1), rather we look for solutions to a specific $y(t)$ which at some time t_0 has the value y_0, or we want to determine a $y(t)$ such that

$$\frac{dy}{dt} + a(t)y = 0, \quad y(t_0) = y_0.$$

Please accept the derivation that a general solution to this type of problem is

$$y(t) = y_0 \exp\left(-\int_{t_0}^{t} a(s)\, ds\right) \tag{2}$$

without proof (there is nothing of interest about the derivation process).

Example 1.4. To solve

$$\frac{dy}{dt} + (\sin t)y = 0, \quad y(0) = \frac{3}{2},$$

let $a(t) = \sin t$, $t_0 = 0$, $y_0 = \frac{3}{2}$. Then

$$y(t) = \frac{3}{2} \exp\left(-\int_0^t \sin s\, ds\right) = \frac{3}{2} \exp\left(\cos t - 1\right).$$

Example 1.5. To solve the initial value problem

$$\frac{dy}{dt} + \exp(t^2)y = 0, \quad y(1) = 2,$$

simply PLUG IT IN (reee) to get

$$y(t) = 2 \exp\left(-\int_1^t e^{s^2}\, ds\right).$$

Recall that this is the *Gaussian Integral* and can be solved by a change to double integration by polar coordinates (yielding $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$), but in general has no closed form solution.

1.4 Non-homogeneous Linear Differential Equations

Recall the non-homogeneous linear differential equation of the form $\frac{dy}{dt} + a(t)y = b(t)$. Let $\mu(t)$ be a continuous function. Then multiply both sides by a continuous function $\mu(t)$ to get

$$\mu(t)\frac{dy}{dt} + a(t)\mu(t)y = \mu(t)b(t),$$

which is equivalent to the above form of a non-homogeneous linear differential equation. What we want: $\mu(t)\frac{dy}{dt} + a(t)\mu(t)y$ equal to the derivative of some simple expression. To get this, notice that

$$\frac{d}{dt}\mu(t)y = \mu(t)\frac{dy}{dt} + \frac{d\mu}{dt}y$$

by the product rule, so if $\frac{d\mu(t)}{dt} = a(t)\mu(t)$, our expression above will simply be equal to the derivative of $\mu(t)y$. Since our new expression is just a linear homogeneous differential equation, we have

$$\mu(t) = \exp\left(\int a(t)\, dt\right).$$

Now the expressions

$$\frac{d}{dt}\mu(t)y = \mu(t)\frac{dy}{dt} + \frac{d\mu}{dt}y$$

and

$$\frac{d}{dt}\mu(t)y = \mu(t)b(t)$$

are equivalent, so we can integrate both sides to obtain

$$\mu(t)y = \int \mu(t)b(t)\,dt + c$$

or

$$y = \frac{1}{\mu(t)}\left(\int \mu(t)b(t)\,dt + c\right) = \exp\left(-\int a(t)\,dt\right)\left(\int \mu(t)b(t)\,dt + c\right).$$

A similar integration between t_0 and t yields

$$\mu(t)y - \mu(t_0)y_0 = \int_{t_0}^{t} \mu(s)b(s)\,ds$$

or

$$y = \frac{1}{\mu(t)}\left(\mu(t_0)y_0 + \int_{t_0}^{t} \mu(s)b(s)\,ds\right),$$

solving initial-value problems.

1.5 Integrating Factor

Remark: $\mu(t)$ is called an *integrating factor* for the nonhomogeneous equation since after multiplying both sides by $\mu(t)$ we can immediately integrate to find all solutions.

Example 1.6. We find the general solution of the differential equation

$$\frac{dy}{dt} - 2ty = t.$$

We know the integrating factor $\mu(t)$ is equal to $\exp\left(\int -2t\,dt\right) = e^{-t^2}$, so multiplying both sides by $\mu(t)$ yields

$$e^{-t^2}\left(\frac{dy}{dt} - 2ty\right) = e^{-t^2}t$$

which is equivalent to

$$\frac{d}{dt}\left(e^{-t^2}y\right) = e^{-t^2}t$$

by our choice of $\mu(t)$. So

$$\int \frac{d}{dt}\left(e^{-t^2}y\right)dt = \int te^{-t^2}\,dt,$$

and by the Fundamental Theorem we have

$$e^{-t^2}y = -\frac{1}{2}e^{-t^2} + c.$$

Finally, we conclude that

$$y = -\frac{1}{2} + ce^{t^2}.$$

Example 1.7. Here we solve an initial value problem. Let

$$\frac{dy}{dx} + xy = xe^{x^2/2}, \quad y(0) = 1.$$

The integrating factor $\mu(x)$ is equal to $\exp\left(\int x\,dx\right) = e^{x^2/2}$. Multiply both sides by $\mu(x)$ to obtain

$$e^{x^2/2}\left(\frac{dy}{dx} + xy\right) = e^{x^2}\left(xe^{x^2/2}\right),$$

which is equivalent to

$$\frac{d}{dx}\left(e^{x^2/2}y\right) = xe^{x^2}$$

by our choice of $\mu(x)$. By the Fundamental Theorem, we have

$$e^{x^2/2}y = \int xe^{x^2}\,dx = \frac{1}{2}e^{x^2} + c.$$

Let $x = 0$ and $y = 1$, then $1 = \frac{1}{2} + c$, so $c = \frac{1}{2}$. We conclude that

$$y(x) = \frac{1}{2}e^{x^2/2} + \frac{1}{2}e^{-x^2/2}.$$

We can simplify this to

$$y(x) = \frac{1}{2}e^{x^2/2}\left(1 + e^{-x^2}\right).$$

Lecture 2

First Test Review

As you can see, I gave up taking notes for this class. It's no fun. I don't care about logistic equations or manually calculating things[1]. If I lived in a world where all I did was proofs, life would be much better. Alas, I have a test in two days, and this is not the case. So, here we are.

We didn't cover some basic stuff that everyone should know (variation of parameters, proving uniqueness-existence, Picard iteration, series solutions), you know, basically what I signed up for this class to learn. So we'll cover those later with proper sections in my free time.

2.1 First-order linear differential equations (homogeneous)

First order linear ODE's are of the form

$$\frac{dy}{dt} + a(t)y = b(t). \tag{3}$$

We solve the homogeneous case, $\frac{dy}{dt} + a(t)y = 0$ by (intuitively) dividing by y and writing $\frac{y'}{y} = \frac{dy}{dt}/y$ as $\frac{d}{dt}\ln|y(t)|$. Then it pretty much immediately follows that

$$y(t) = \exp\left(-\int a(t)\,dt\right).$$

2.2 Initial value problem homogeneous 1st order ODE

Above gives solution sets of infinite order. Sometimes engineers care about initial value problems, that is, we want to solve equations of the form

$$\frac{dy}{dt} + a(t)y = 0, \quad y(t_0) = y_0. \tag{4}$$

If we just follow the same steps as earlier and integrate with bounds, we get

$$y(t) = y_0 \exp\left(-\int_{t_0}^{t} a(s)\,ds\right).$$

2.3 Nonhomogeneous linear 1st order ODEs

They are of the form

$$\frac{dy}{dt} + a(t)y = b(t). \tag{5}$$

Multiply by a continuous $\mu(t)$ such that we have $\frac{dy}{dy}\mu(t) + a(t)\mu(t)y = \mu(t)b(t)$: if $\frac{d}{dt}\mu(t)y = \frac{d\mu}{dt}y + \frac{dy}{dt}\mu$, then simply replace the left half of the expression with this, and notice that they're equal if $\frac{d\mu(t)}{dt} = a(t)\mu(t)$. So $\mu(t) = c\exp\left(\int a(t)\,dt\right)$. Therefore we have

$$\frac{d}{dt}\mu(t)y = \mu(t)b(t) \implies y = \frac{1}{\mu(t)}\left(\int \mu(t)b(t)\,dt + c\right),$$

which is the general solution.

[1] This was a horrible premonition...

2.4 Initial value nonhomogeneous linear 1st order ODE

We're given something that looks like

$$\frac{dy}{dt} + a(t)y = b(t), \quad y(t_0) = y_0. \tag{6}$$

To solve this, literally just integrate on the bounds. We get that solutions are of the form

$$y = \frac{1}{\mu(t)}\left(y_0\mu(t_0) + \int_{t_0}^{t} \mu(s)b(s)\,ds\right).$$

2.5 Separable equations

They are of the form

$$\frac{dy}{dx} = g(x)f(y). \tag{7}$$

Because you can just do this: $\frac{dy}{dx} = \frac{g(x)}{h(y)}$, where $h = f^{-1}$ given $f \neq 0$ on its domain. Nobody knows what a differential form actually is, but it's apparent how to solve it (nonrigorously).

2.6 The logistic equation

I hope this doesn't show up or I'm gonna lose my mind.

$$p(t) = \frac{ap_0}{bp_0 + (a - bp_0)e^{-a(t-t_0)}} \tag{8}$$

This comes from $\frac{dp}{dt} = ap - bp^2$. The solution to an IVP with $p(t_0) = p_0$ is $a(t - t_0) = \ln \frac{p}{p_0}\left|\frac{a_0 b p_0}{a - bp}\right|$.

2.7 Second order linear homogenous differential equations

They are of the form

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0. \tag{9}$$

By the existence-uniqueness theorem, there exists a unique solution $y(t)$ satisfying this ODE on an open interval (with given initial conditions $y(t_0) = y_0$, $y'(t_0) = y_0'$). Let's define an operator by

$$L[y](t) = y''(t) + p(t)y'(t) + q(t)y(t).$$

This is just a natural transformation if we view maps as functors from $\mathbb{R}$ to $\mathbb{R}$ (category theory ftw). If $L[cy] = cL[y]$ and $L[y_1 + y_2] = L[y_1] + L[y_2]$ for $c \in \mathbb{R}$, $y_1, y_2 \colon \mathbb{R} \to \mathbb{R}$, we say L is a *linear operator*. You can verify that $L[y](t)$ defined above is linear. Clearly just solve for $L[y](t)$ and we get the solutions to the second-order ODE. Here's the useful thing: by this fact, we get that

$$c_1 y_1(t) + c_2 y_2(t)$$

is the general form of solutions to Equation (9), where $c_1, c_2 \in \mathbb{R}$ and y_1, y_2 are particular solutions to Equation (9). You can see this by evaluating $L[c_1 y_1(t) + c_2 y_2(t)]$ and applying linearity properties. In particular, *all* solutions to Equation (9) are of that form, by a quick application of the existence uniqueness theorem, given that the gradient vectors are linearly independent (checking this is just a quick calculation to see that the Wronskian is nonzero). We say $\{y_1, y_2\}$ is a *fundamental set* of solutions of Equation (9).

2.8 Second order homogeneous ODE constant coefficients

General method for constant coefficients: let's say they're of the form

$$L[y] = a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0, \tag{10}$$

where a, b, c are constants and a nonzero. Then just look at the characteristic polynomial $P(r) = ar^2 + br + c$, and examine the roots r_1, r_2 such that $(r - r_1)(r - r_2) = 0$. If $r_1 \neq r_2,\ r_1, r_2 \in \mathbb{R}$, then $e^{r_1 x}, e^{r_2 x}$ are LI solutions to Equation (10) so the general solution is of the form

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

If $r_1 = r_2 = r,\ r_1, r_2 \in \mathbb{R}$, then e^{rx}, xe^{rx} are LI solutions and the general solution is of the form

$$y = c_1 e^{rx} + c_2 x e^{rx}.$$

Finally, if $r_1 \in \mathbb{C}$ (that is, $r_1 = a + bi$) for $a, b \in \mathbb{R}$), then r_2 is the complex conjugate of r_1 (that is, $r_2 = \overline{r_1} = a - bi$) and the functions $e^{ax}\cos(bx)$, $e^{ax}\sin(bx)$ are LI solutions to Equation (10) and the general solution is of the form

$$y = c_1 e^{ax}\cos(bx) + c_2 e^{ax}\sin(bx).$$

2.9 Nonhomogeneous second order ODEs

Let's turn our attention to the big boy, the nonhomogeneous second order differential equation given by

$$L[y] = \frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t), \tag{11}$$

where the functions $p(t), q(t)$ and $g(t)$ are continuous on an open interval.

Theorem 2.1. *Every solution of Equation (11) is of the form*

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \psi(t)$$

where y_1, y_2 are LI solutions to Equation (9), $\psi(t)$ is a particular solution to Equation (11), and c_1, c_2 are constants.

Proof. We need a lemma.

Lemma 2.1. *The difference of any two solutions of Equation (11) is a solution of Equation (9).*

Proof. If y_1, y_2 are two solutions of Equation (11), then $L[y_1 - y_2] = L[y_1] - L[y_2] = g(t) - g(t) = 0$. ⊠

Now returning to the proof of the theorem, we know $y(t)$ is a solution of Equation (11) by definition. Then by Lemma 2.1, $\phi(t) = y(t) - \psi(t)$ is a solution of Equation (9). But since every solution of Equation (9) is of the form $c_1 y_1(t) + c_2 y_2(t)$, we have

$$y(t) = \phi(t) = \psi(t) = c_1 y_1(t) + c_2 y_2(t) + \psi(t).$$

⊠

2.10 The method of judicious guessing

Is this the actual name of the method? We try to guess solutions for equations of the form

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = g(t), \tag{12}$$

where $a, b, c \in \mathbb{R}$ and $g(t)$ is of a certain form, described below. **Case 1:** The differential equation is of the form

$$L[y] = a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = a_0 + a_1 t + \cdots + a_n t^n.$$

It can be shown that there is a solution of the form

$$\psi(t) = \begin{cases} A_0 + A_1 t + \cdots + A_n t^n, & c \neq 0, \\ t(A_0 + A_1 t + \cdots + A_n t^n), & c = 0,\ b \neq 0, \\ t^2(A_0 + A_1 t + \cdots + A_n t^n), & c = b = 0. \end{cases}$$

Case 2: The differential equation is of the form

$$L[y] = a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = (a_0 + a_1 t + \cdots + a_n t^n)e^{\alpha t}.$$

Then it can be shown that there is a particular solution of the form

$$\psi(t) = \begin{cases} (A_0 + A_1 t + \cdots + A_n t^n)e^{\alpha t}, & \text{if } e^{\alpha t} \text{ is not a solution of the homogeneous equation,} \\ t(A_0 + A_1 t + \cdots + A_n t^n)e^{\alpha t}, & \text{if } e^{\alpha t} \text{ is a solution of the homogeneous equation, but } te^{\alpha t} \text{ is not,} \\ t^2(A_0 + A_1 t + \cdots + A_n t^n)e^{\alpha t}, & \text{if } e^{\alpha t} \text{ and } te^{\alpha t} \text{ are both solutions of the homogeneous equation.} \end{cases}$$

Equivalently, we have

$$\psi(t) = \begin{cases} (A_0 + A_1 t + \cdots + A_n t^n)e^{\alpha t}, & \text{if } \alpha \text{ is not a solution of the characteristic equation,} \\ t(A_0 + A_1 t + \cdots + A_n t^n)e^{\alpha t}, & \text{if } \alpha \text{ is one of two distinct solutions of the characteristic,} \\ t^2(A_0 + A_1 t + \cdots + A_n t^n)e^{\alpha t}, & \text{if } \alpha \text{ is the only solution of the characteristic equation.} \end{cases}$$

Case 3: Let $\phi(t) = u(t) + iv(t)$ be a particular solution of

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = (a_0 + a_1 t + \cdots + a_n t^n)e^{i\omega t}.$$

All you have to do is look at the real and imaginary parts to get $\mathrm{Re}(\phi(t)) = u(t)$ a solution of $ay'' + by' + cy = (a_0 + a_1 t + \cdots + a_n t^n)\cos(\omega t)$, and similarly $\mathrm{Im}(\phi(t)) = v(t)$ a solution of $ay'' + by' + cy = (a_0 + a_1 t + \cdots + a_n t^n)\sin(\omega t)$.

Remark 2.1. To find solutions where the function on the right is of the form $e^{2t} + e^{-3t}$ or $t \sin t + e^t$ or something like that, simply find solutions to the two componenents and add them.

2.11 Mechanical Vibrations

Equilibrium, spring has been stretched a distance of Δl, where $k\Delta l = mg$. $y(t)$ is position of the mass at time t. To compute this, we must find the total force acting on m, the sum of four forces W, R, D, F.

(i) $W = mg$ is the weight.

(ii) R is the restoring force, given by $R = -k(\Delta l + y)$.

(iii) D is the damping or drag force exerted by the medium (oil, etc) on m, given by $D = -c\frac{dy}{dt}$.

(iv) F denotes the external forces, usually dependent only on time.

So we have

$$m\frac{d^2y}{dt^2} = W + R + D + F = mg - k(\Delta l + y) - c\frac{dy}{dt} + F(t) = -ky - c\frac{dy}{dt} + F(t).$$

So $y(t)$ satisfies the second order ODE given by

$$m\frac{d^2y}{dt^2} + c\frac{dt}{dt} + ky = F(t).$$

(a) *Free vibrations:* In the case of free undampened motion, the differential equation reduces to $m\frac{d^2y}{dt^2}+ky=0$ or $\frac{d^2y}{dt^2}+\omega_0^2y=0$ with $\omega_0^2 := \frac{k}{m}$. The characteristic is $r^2=-\omega_0^2$, so $r=0\pm\omega_0 i$, and the general solution is of the form $y(t)=a\cos(\omega_0 t)+b\sin(\omega_0 t)$. This can be rewritten as $y(t)=R\cos(\omega_0 t-\delta)$, where $R=\sqrt{a^2+b^2}$ and $\delta=\tan^{-1}\left(\frac{b}{a}\right)$. The motion of the mass is periodic with period $T_0=2\pi/\omega_0$, this is known as simple harmonic motion. R is the amplitude, δ the phase angle, and $\omega_0=\sqrt{k/m}$ the natural frequency.

(b) whatever

Lecture 3

Examples

3.1 Homogeneous 1st order ODE

Problem. *Find the general solution of*

$$\frac{dy}{dt}+2ty=0.$$

Solution. $y=c\exp\left(-\int 2t\,dt\right)=c\exp\left(-t^2\right)$. ■

3.2 Homogeneous first order ODE initial value

Problem. *Find the solution of*

$$\frac{dy}{dt}+(\sin t)y=0, \quad y(0)=\frac{3}{2}.$$

Solution. $y=\frac{3}{2}\exp\left(-\int_0^t \sin t\,dt\right)=\frac{3}{2}\exp(\cos t-1)$. ■

Problem. *Solve*

$$\frac{dy}{dt}+e^{t^2}y=0, \quad y(1)=2.$$

Solution. $y=2\exp\left(-\int_1^t e^{t^2}\,dt\right)$. This function isn't integrable (to be precise, no closed form solution exists) so we're done. ■

3.3 Nonhomogeneous first order

Problem. *Solve*

$$\frac{dy}{dt}-2ty=t.$$

Solution. Let $\mu(t)=\exp\left(\int -2t\,dt\right)=\exp\left(-t^2\right)$. So $\frac{d}{dt}y\cdot\exp\left(-t^2\right)=\exp\left(-t^2\right)t \implies y\cdot\exp(-t^2)=-\frac{1}{2}e^{-t^2}+c \implies y=-\frac{1}{2}+ce^{t^2}$. ■

Problem. *Solve*

$$x\frac{dy}{dx}+y=\cos x,\ x>0.$$

Solution. We have $\frac{dy}{dx}+\frac{y}{x}=\frac{\cos x}{x}$. So $\mu(x)=e^{|\ln(x)|}=x$ for all x strictly positive. Then $\frac{d}{dx}yx=x\frac{\cos x}{x}=\cos x$. So $xy=\sin x+c \implies y=\frac{\sin x}{x}+\frac{c}{x}$. ■

3.4 Nonhomogeneous first order initial value

Problem. *Solve*

$$\frac{dy}{dt} + 2ty = t, \quad y(1) = 2.$$

Solution. We have $\mu(t) = e^{t^2}$. So $\frac{d}{dt} y e^{t^2} = t e^{t^2} \implies y e^{t^2} = \frac{1}{2} e^{t^2} + c \implies y = \frac{1}{2} + c e^{-t^2}$. At $y(1) = 2$, we have $\frac{3}{2} = \frac{c}{e} \implies c = \frac{3}{2} e$. So the solution is $y = \frac{1}{2} + \frac{3}{2} e^{(-t^2+1)}$. ■

Problem. *Solve*

$$\frac{dy}{dx} + xy = x e^{\frac{x^2}{2}}, \quad y(0) = 1.$$

Solution. Now $\mu(t) = e^{\frac{x^2}{2}}$. So $\frac{d}{dx} y e^{\frac{x^2}{2}} = x e^{x^2}$, and $y e^{\frac{x^2}{2}} = \frac{1}{2} e^{x^2} + c \implies y = \frac{1}{2} e^{\frac{x^2}{2}} + c e^{-\frac{x^2}{2}}$. At $y(0) = 1$, we have $1 = \frac{1}{2} + c$, so $c = \frac{1}{2}$, and the general solution is of the form $y = \frac{1}{2} e^{\frac{x^2}{2}} + \frac{1}{2} e^{-\frac{x^2}{2}} = \frac{1}{2} e^{\frac{x^2}{2}} \left(1 + e^{-x^2}\right)$. ■

3.5 Separable equations

Problem. *Solve*

$$\frac{dy}{dx} = \frac{x^2}{y^2}.$$

Solution. $y^3 = x^3 + c \implies y = \sqrt[3]{x^3 + 3c}$. ■

Problem. *Solve*

$$\frac{dy}{dx} = \frac{6x^2}{2y + \cos y}.$$

Solution. $\int 2y + \cos y \, dy = 2x^3 \implies y^2 + \sin y + c = 2x^3$. Now what? I think it's over. ■

Problem. *Solve*

$$y' = x^2 y.$$

Solution. $\ln|y| = \frac{x^3}{3} + c \implies y = \pm e^{\frac{x^3}{3} + c} \implies y = C e^{\frac{x^3}{3}}$. ■

3.6 Logistic equations

Example 3.1. Suppose the population at time t satisfies the IVP

$$\frac{dp}{dt} = p - \frac{1}{100} p^2, \; p(t_0) = p_0,$$

where p_0 is the population at time t_0. Then $t - t_0 = \ln \frac{p(100 - p_0)}{p_0(100 - p)}$.

3.7 Second order homogeneous ODEs

Problem. *Find the solutions of*

$$\frac{d^2 y}{dt^2} + y = 0.$$

Solution. Clearly two particular solutions are $y_1(t) = \cos t$, $y_2(t) = \sin t$, then by the existence uniqueness thm the general solution is of the form $y(t) = c_1 \cos t + c_2 \sin t$. ■

Problem. *Calculate the Wronskian for y_1, y_2.*

Solution. Why am I doing this??? I have better things to do. ■

3.8 Second order ODE constant coefficients

Problem. *Determine all solutions to the differential equation*

$$y'' + y' - 6y = 0$$

of the form e^{rx}.

Solution. $y' = re^{rx}, y'' = r^2e^{rx}$. So we have $e^{rx}(r^2 + r - 6) = 0$ for the differential equation. Clearly $r = 2, -3$ satisfy this equation, so the solutions are $y_1 = e^{2x}, y_2 = e^{-3x}$. These are LI, so the general solution is of the form $c_1e^{2x} + c_2e^{-3x}$. ■

Problem. *Solve*

$$y'' + y = 0.$$

Solution. The characteristic is $r^2 + 1$, so $r_1 = i$ and $r_2 = -i$. Then solutions are of the form $c_1e^0\cos(1x) + c_2e^0\sin(1x) = c_1\cos x + c_2\sin x$. ■

Problem. *Solve*

$$y'' + 6y' + 25y = 0.$$

Solution. The solutions to the characteristic polynomial $r^2+6r+25$ are simply $r = -3\pm4i$. So the general solution is of the form $c_1e^{-3x}\cos(4x) + c_2e^{-3x}\sin(4x)$. ■

Problem. *Solve the following initial value problem:*

$$y'' + 4y' + 4y = 0, \quad y(0) = 1, \, y'(0) = 4.$$

Solution. Clearly the general solution is of the form $c_1e^{-2x} + c_2xe^{-2x}$. At $y(0) = 1$, we have $1 = c_1$. Then $y' = -2e^{-2x} + c_2e^{-2x} + -2c_2xe^{-2x}$, so at $y'(0) = 4$ we have $4 = -2 + c_2$. So $c_2 = 6$, and the general solution is of the form $e^{-2x} + 6xe^{-2x}$. ■

3.9 Nonhomogeneous second order ODEs

Problem. *Three solutions of some second-order nonhomogeneous ODE are*

$$\varphi_1(t) = t, \varphi_2(t) = t + e^t, \textit{ and } \varphi_3(t) = 1 + t + e^t.$$

Find the general solution of the equation.

Solution. By our lemma, $\varphi_2 - \varphi_1 = e^t$ and $\varphi_3 - \varphi_2 = 1$ are clearly LI solutions to the nonhomogeneous equation. Then the general solution is of the form $c_1 + c_2e^t + t$. Our choices of φ_i don't really matter, just trust the theorems. ■

Problem. *Three solutions of some second order nonhomogeneous linear ODE are*

$$\phi_1(t) = t^2, \, \phi_2(t) = t^2 + e^{2t}, \textit{ and } \phi_3(t) = 1 + t^2 + 2e^{2t}.$$

Find the general solution of the equation.

Solution. Just take the difference of two, it'll work out. ■

3.10 Judicious guessing (nonhomogeneous second order with constant coefficients)

Problem. *Find a particular solution* $\psi(t)$ *of the equation*

$$L[y] = \frac{d^2y}{dt^2} + \frac{dy}{dt} + y = t^2.$$

Solution. Since $c \neq 0$, we have a solution $\psi(t)$ of the form $A_0 + A_1t + A_2t^2$. Plug that in and solve for the constants. ■

Problem. *Find a particular solution*

3.11 Mechanical Vibrations

Problem. *A* 1 *kg mass stretches a spring 49/320 m. Pull down the mass an additional 1/4, find amplitude, period, and frequency.*

Solution. We have $m = 1$ kg, $g = 9.8$ m/s^2, and $\Delta l = \frac{49}{320}$ m. Since $k\Delta l = mg$, we have $k = 9.8 \cdot \frac{320}{49} \approx 64$, and $\omega_0 \approx \sqrt{64} \approx 8$. So $y(t) = a\cos(8t) + b\sin(8t)$. The initial position is $\frac{1}{4}$ m, so $y(0) = \frac{1}{4}$ and $a = \frac{1}{4}$. Consider $y'(t) = -8a\sin(8t) + 8b\cos(8t)$, but $y'(0) = 0$ so $b = 0$. Therefore $y = \frac{1}{4}\cos(8t)$, the amplitude is $\frac{1}{4}$, period is $\frac{2\pi}{8} = \frac{\pi}{4}$, and frequency is 8. ∎

Lecture 4

Second Test Review

I really wish I had an RREF or eigen-blah calculator for the test.

4.1 Algebraic Properties of Solutions of Linear Systems

You can convert any n-th order differential equation to a system of n first order differential equations. Just let

$$x_1 = y,\ x_2 = y',\ x_3 = y'',\cdots,\ x_n = y^{(n-1)}$$

and everything will work out. Yes, you can write things as matrices. The rest of the section just goes on and on about manually plugging stuff in and solving by methods from previous sections.

4.2 Vector Spaces

If anybody wasn't annoyed enough already by the ridiculous memorization/plug and chug approach to computational lower level mathematics, here are ten axioms you should memorize about vector spaces. For $u, v, w \in V$ and $a, b, c \in \mathbb{F}$ we have the following:

(i) $u + v \in V$

(ii) $cu \in V$

(iii) $u + v = v + u$

(iv) $(u + v) + w = u + (v + w)$

(v) $\exists 0 \in V \ni 0 + u = u$

(vi) $\forall u \in V\ \exists(-u) \in V \ni u + (-u) = 0$

(vii) $1 \cdot u = u$

(viii) $a(bu) = (ab)u$

(ix) $a(u + v) = au + av$

(x) $(a + b)u = au + bu$

For example, $\mathbb{R}^n$, $\mathbb{P}^n, \mathbb{R}, \mathbb{P}, \mathbb{R}^{\mathbb{R}}, \{0\}, \mathrm{GL}_n(\mathbb{F})$ etc are all vector spaces. You know what a subspace is. And clearly subspaces form vector spaces themselves. You can show weird things are spaces by considering them as subspaces of $\mathbb{R}^n$. Also, linear combinations of vectors of a space form a subspace (and therefore a space).

4.3 Dimension of a Vector Space

Definition 4.1 (Linear independence and dependence). Vectors $v_1, \cdots, v_p$ are **linearly dependent** if there exist scalars $c_1, \cdots, c_p$ not all zero such that

$$c_1 v_1 + \cdots + c_p v_p = 0. \tag{13}$$

If the only solution to Equation (13) is the trivial one, that is, $c_i = 0$ for all i, then the vectors are said to be **linearly independent**.

To show things are LI or not, just reduce them: if there's a zero column, then dep, if you can get it into REF, then LI. This is the first method you learn. Note that vectors aren't LI iff one is a linear combo of the others, if there are only two this is equivalent to one being a scalar multiple of the other.

Definition 4.2 (Dimension). The **dimension** of a vector space V, denoted by $\dim V$, is the order of any basis for V. Note that we can have zero dimensional vector spaces, for example take the trivial space $\{0\}$.

4.4 Applications of Linear Algebra to Differential Equations

Theorem 4.1 (Existence-uniqueness). *There exists exactly one solution to the IVP*

$$\dot{x} = Ax, \quad x(t_0) = x^{(0)} = \begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \\ \vdots \\ x_n^{(0)} \end{bmatrix}.$$

Theorem 4.2. *Let $x^{(1)}, x^{(2)}, \cdots, x^{(k)}$ be k solutions for $\dot{x} = Ax$. Then for some t_0 we have $x^{(1)}, \cdots, x^{(k)}$ LI solutions iff $x^{(1)}(t_0), \cdots, x^{(k)}(t_0)$ are LI vectors in $\mathbb{R}^n$.*

Remark 4.1. There is no square matrix A with constant entries such that $x^{(1)}(t)$, $x^{(2)}(t)$ are solutions of $\dot{x} = Ax$.

4.5 The Theory of Determinants

Do I even need to take notes? Recall that if A is triangular then we can just multiply along the diagonal (be lazy! –Dr. Tran). Swapping two rows (WLOG, since for columns note that $\det A = \det A^T$) gives a negative determinant, multiplying rows by a scalar k gives $k \det A$, adding two rows does nothing.

4.6 Solutions of Simultaneous Linear Equations

Do you know how to multiply matrices? Do you know about noncommutative rings? (OHO big scary) Do you know that cancellation only holds in integral domains (which $M_{n\times n}(\mathbb{R})$ isn't due to the existence of zero divisors)? OK good. Also, can you invert matrices?

Theorem 4.3 (Cramer's rule). *Let A be an invertible $n \times n$ matrix. For any $b \in \mathbb{R}^n$, the unique solution x of $Ax = b$ has entries given by*

$$x_i = \frac{\det A_i(b)}{\det A}, \quad i \in \mathbb{N}.$$

To make sense of this, imagine replacing the column vector representing x_i in the matrix with the vector b and taking its determinant, then dividing by $\det A$.

4.7 Linear Transformations

If you've heard of 3Blue1Brown, then you probably know that matrices just represent linear transformations (maps) $\mathbb{R}^n \to \mathbb{R}^m$. Hence the name, transformations are linear, so $T(u+v) = T(u) + T(v)$ and $T(cu) = cT(u)$ for all $u, v \in V$ and c scalars.

Theorem 4.4. *The equation $Ax = b$ has a unique solution if the columns of A are LI, and has either no solution or infinitely many solutions if the columns of A are linearly dependent.*

Elementary row operations are adding a multiple of a row to another, multiplying a row by a nonzero constant, and interchanging two rows. REF is where zero rows are at the bottom, leading entries are to the right of the leading entry of the row above it, and all entries in a column below a leading entry are zero. RREF is when the leading entries are all pivots (1), and each leading 1 is the only nonzero entry in its column.

Definition 4.3. A **pivot position** in a matrix is a location in A that corresponds to a leading 1 in the RREF of A. A pivot column is a column that contains a pivot, variables that correspond to pivot columns are **basic variables**, while the other variables are **free variables**. A system having free variables means infinitely many solutions.

Lemma 4.1. *The columns of an $n \times n$ matrix are LI iff $\det A \neq 0$.*

Proof. Here's a neat proof.

(1) The cols of A are LI iff $Ax = b$ has a unique solution for all $b \in \mathbb{R}^n$ by Theorem 4.4.

(2) The equation $Ax = b$ has a unique solution x for all b iff the linear transformation $T(x) = Ax$ has an inverse.

(3) This this is true iff A^{-1} exists.

(4) Which is true iff $\det A \neq 0$.

⊠

Theorem 4.5. *The equation $Ax = b$ has a unique solution $x = A^{-1}b$ if $\det A \neq 0$. $Ax = b$ has either no solutions or infinitely many solutions if $\det A = 0$.*

Corollary 4.1. *The equation $Ax = 0$ has a nontrivial solution iff $\det A = 0$.*

4.8 The Eigenvalue-Eigenvector Method of Finding Solutions

Definition 4.4 (Eigenblah)**.** An **eigenvector** of an $n \times n$ matrix is a nonzero vector x such that $Ax = \lambda x$ for some scalar λ, we say λ is an **eigenvalue** of A. The set of all solutions of the equation $(A - \lambda I) = 0$ is the **eigenspace** of A corresponding to λ. $\det(A - \lambda I)$ is the **characteristic polynomial** of A and $\det(A - \lambda I) = 0$ is the **characteristic equation** of A.

For $\dot{x} = Ax$, we want to find n LI solutions $x^1(t), \cdots, x^n(t)$.

Theorem 4.6. *We have $x(t) = e^{\lambda t}v$ a solution of $\dot{x} = Ax$ iff λ is an eigenvalue and v is an eigenvector of A.*

Proof. Exponential functions are invariant under reduction, so let's make an educated guess that $x(t) = e^{\lambda t}v$ is a solution. Since $\frac{d}{dt}e^{\lambda t}v = \lambda e^{\lambda t}v$ and $A(e^{\lambda t}v) = e^{\lambda t}Av$, we have $\dot{x} = \lambda e^{\lambda t}v = Ax = e^{\lambda t}Av$, so $Av = \lambda v$, hence $x(t) = e^{\lambda t}v$ is a solution of $\dot{x} = Ax$ iff λ is an eigenvalue and v is an eigenvector of A. ⊠

4.9 Complex Roots

If $\lambda = \alpha + i\beta$ is a complex eigenvalue of A with eigenvector $v = v^1 + iv^2$, then $x(t) = e^{\lambda t}v$ is a complex valued solution of the differential equation $\dot{x} = Ax$.

Lemma 4.2. *Let $z(t) = x(t) + iy(t)$ be a complex valued solution of $\dot{x} = Ax$. Then $\operatorname{Re} z$ and $\operatorname{Im} z$ (AKA $x(t)$ and $y(t)$) are both real valued solutions of $\dot{x} = Ax$, and are LI.*

4.10 Equal Roots

What if the characteristic has a root with multiplicity greater than one? Suppose $A_{n\times n}$ has $k < n$ LI eigenvectors, then the differential equation $\dot{x} = Ax$ only has k LI solutions of the form $e^{\lambda t}$, and our goal is to find $n - k$ more LI solutions. This is how we'll do it: recall that $x(t) = e^{at}c$ is a solution of the scalar differential equation $\dot{x} = ax$ for every constant c. What we want: $x(t) = e^{At}v$ a solution for $\dot{x} = Ax$ for all constant vectors v. But e^{At} isn't defined if A is a matrix, turns out this isn't too bad of an issue. Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, define $e^{At} := I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots + \frac{A^nt^n}{n!} + \cdots$. This converges, and we can differentiate termwise to get

$$\frac{d}{dt}e^{At} = A = A^2 + \frac{A^3t^2}{2!} + \cdots + \frac{A^{n+1}t^n}{n!} + \cdots = Ae^{At},$$

which implies that $e^{At}v$ is a solution of $\dot{x} = Ax$ for every constant vector v, since $\frac{d}{dt}e^{At}v = Ae^{At}v = A(e^{At}v)$. Warning! $e^{At+Bt} = e^{At}e^{Bt} \iff AB = BA$. This leads us to our general algorithm for finding more solutions:

(1) Find all eigenvalues and eigenvectors of A: if there are n of them, we are done.

(2) If we only found $k < n$ eigenvalues and eigenvectors, then to find additional solutions, we pick an eigenvalue λ of A and find all vectors v such that $(A - \lambda I)^2 v = 0, \quad$ but $(A - \lambda I)v \neq 0$. For each such vector v,

$$e^{At}v = e^{\lambda t}e^{(A-\lambda I)t} = e^{\lambda t}\left[v + t(A - \lambda I)v\right]$$

is an additional solution of $\dot{x} = Ax$. The equation holds since $e^{At}v = e^{(A-\lambda I+\lambda I)t}v = e^{\lambda t}e^{(A-\lambda I)t}v$, which is true because the matrices commute. We do this for all eigenvalues of A.

(3) If we still don't have enough solutions, then we find all vectors v such that $(A-\lambda I)v^3 = 0$, but $(A-\lambda I)v^2 \neq 0$. For each such v,

$$e^{At}v = e^{\lambda t}\left[v + t(A-\lambda I)v + \frac{t^2}{2!}(A-\lambda I)^2 v\right]$$

is an additional solution of $\dot{x} = Ax$. The reason why this algorithm works is because if $(A-\lambda I)^m = 0$, then the series $e^{(A-\lambda I)t}v$ terminates after m terms. Indeed, if $(A-\lambda I)^m = 0$, then $(A-\lambda I)^{m+\ell}v = 0$ as well, since $(A-\lambda I)^{m+\ell}v = (A-\lambda I)^{\ell}\left[(A-\lambda I)^m v\right] = 0$. As a consequence, $e^{(A-\lambda I)t}v = v + t(A-\lambda I)v + \frac{t^2}{2!}(A-\lambda I)^2 v + \cdots + \frac{t^{m-1}}{(m-1)!}(A-\lambda I)^m v$, which implies that

$$e^{At}v = e^{\lambda} = e^{\lambda t}e^{(A-\lambda I)t}v = e^{\lambda t}\left[v + t(A-\lambda I)v + \frac{t^2}{2!}(A-\lambda I)^2 v + \cdots + \frac{t^{m-1}}{(m-1)!}(A-\lambda I)^{m-1}v\right].$$

(4) We continue in this manner until, hopefully, we obtain n LI solutions.

4.11 Fundamental Matrix Solutions; e^{At}

If $x^1(t), \cdots, x_n(t)$ are n LI solutions of the differential equation $\dot{x} = Ax$, then every solution $x(t)$ can be written in the form $x(t) = c_1x^1(t) + c_2x^2(t) + \cdots + c_nx^n(t)$. Let $X(t)$ be the matrix whose columns are $x^1(t), \cdots, x^n(t)$. Then we can write the previous equation in the form $x(t) = X(t)c$, where $c = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$.

Definition 4.5 (Fundamental matrix). A matrix $X(t)$ is called a **fundamental matrix solution** of $\dot{x} = Ax$ if its columns form a set of n LI solutions.

Theorem 4.7. *Let $X(t)$ be a fundamental matrix solution of the differential equation $\dot{x} = Ax$. Then $e^{At} = X(t)X^{-1}(0)$. In other words, the product of any fundamental solution with its inverse at $t = 0$ gives e^{At}.*

Lemma 4.3. *A matrix $X(t)$ is a fundamental matrix solution of $\dot{x} = Ax$ iff $\dot{X}(t) = AX(t)$ and $\det X(0) \neq 0$. We can rewrite the first condition to say that $A = \dot{X}(t)X^{-1}(t)$.*

Lecture 5

Examples

5.1 Algebraic Properties of Solutions of Linear Systems

Problem. *Convert the following differential equation into a system of two first order differential equations:*

$$4\frac{d^2y}{dt^2} + \frac{dy}{dt} + 3y = 0$$

Solution. Let $x_1 = y$, $x_2 = y'$. Then $x_1' = x_2$ and $x_2' = \frac{-x_2-3x_1}{4}$. ■

Problem. *Convert the following IVP.*

$$y''' + (y')^2 + 3y = e^t; \quad y(0) = 1,\ y'(0) = 0,\ y''(0) = 0.$$

Solution. Let $x_1 = y$, $x_2 = y'$, $x_3 = y''$. Then $x_1' = x_2$, $x_2' = x_3$, $x_3' = e^t - x_2^2 - 3x_1$, given $x_1(0) = 1$, $x_2(0) = 0$, $x_3(0) = 0$. ■

5.2 Vector Spaces

Problem. *For a, b scalars, show that the set of all matrices H of the form below is a vector space.*

$$\begin{bmatrix} 4a-b \\ 2b \\ a-2b \\ a-b \end{bmatrix}$$

Solution.

$$\begin{bmatrix} 4a-b \\ 2b \\ a-2b \\ a-b \end{bmatrix} = a\begin{bmatrix} 4 \\ 0 \\ 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} -1 \\ 2 \\ -2 \\ -1 \end{bmatrix},$$

so for all $h \in H$, h is a linear combination of vectors in $\mathbb{R}^4$ and is therefore a vector space. ■

5.3 Dimension of a Vector Space

Example 5.1. Let V be the set of all solutions of the differential equation

$$\frac{d^2x}{dt^2} - x = 0.$$

Since solutions are of the form $x(t) = c_1e^t + c_2e^{-t}$, we have that $x_1(t) = e^t$ and $x_2(t) = e^{-t}$ span V. Note that $\dim(V) = 2$.

5.4 Applications of Linear Algebra to Differential Equations

Example 5.2. The vectors $x^{(1)}(t) = \begin{bmatrix} e^t \\ -3e^t/2 \end{bmatrix}$ and $x^{(2)}(t) = \begin{bmatrix} e^{5t} \\ -e^{5t}/2 \end{bmatrix}$ are LI since $x^{(1)}(0) = \begin{bmatrix} 1 \\ -3/2 \end{bmatrix}$ and $x^{(2)}(0) = \begin{bmatrix} 1 \\ -1/2 \end{bmatrix}$, which are not scalar multiples of each other.

5.5 The Theory of Determinants

zzz

5.6 Solutions of Simultaneous Linear Equations

Example 5.3. To solve $Ax = b$ where $A = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $b = \begin{bmatrix} 1 \\ -7 \end{bmatrix}$, by Cramer's rule we have

$$x_1 = \frac{\begin{vmatrix} 1 & -2 \\ -7 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & -2 \\ 3 & 4 \end{vmatrix}} = \frac{-10}{10} = -1, \quad x_2 = \frac{\begin{vmatrix} 1 & 1 \\ 3 & -7 \end{vmatrix}}{\begin{vmatrix} 1 & -2 \\ 3 & 4 \end{vmatrix}} = -\frac{10}{10} = -1 \implies x = \begin{bmatrix} -1 \\ -1 \end{bmatrix}.$$

5.7 Linear Transformations

Example 5.4. You can show transformations are linear by encoding them with a matrix, for example, considering the mapping $T : x \mapsto \begin{bmatrix} x_1 + x_2 + x_3 \\ x_1 + x_2 - x_3 \\ x_1 \end{bmatrix}$, we show this is linear by noting that the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix}$ encodes this transformation. Also, the standard matrix for clockwise rotation about the origin for an angle ϕ is given by $A = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$.

Problem. *When does the equation $Ax = 0$ have a nontrivial solution?*

$$A = \begin{bmatrix} 1 & \lambda & \lambda \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Solution. We want $\det A = 0$, since if not A would be invertible and so would uniquely have the trivial solution. So $\lambda = 1$. ∎

5.8 The Eigenvalue-Eigenvector Method of Finding Solutions

Example 5.5. To find the general solution of $\dot{x} = Ax$ where $A = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix}$, note that $p(\lambda) = \lambda^2 - 6\lambda + 5$, so the eigenvalues are $\lambda = 1$ and $\lambda = 5$. The corresponding eigenvectors are then $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$, therefore by Theorem 4.6 they're solutions, and since any LC of solutions are also solutions, we have that the solutions are of the form

$$x(t) = c_1 e^t \begin{bmatrix} -2 \\ 3 \end{bmatrix} + c_2 e^{5t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2c_1 e^t - 2c_2 e^{5t} \\ 3c_1 e^t + c_2 e^{5t} \end{bmatrix}.$$

5.9 Complex Roots

Example 5.6. To solve the IVP $\dot{x} = Ax$, where $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$ and $x(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, note that the characteristic is $(1-\lambda)(\lambda^2 - \lambda + 2)$, and so the eigenvalues are $\lambda = 1, 1 \pm i$. For $\lambda = 1$, the corresponding eigenvector is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, and so $x^1(t) = e^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is a solution of the differential equation. For $\lambda = 1 + i$, an eigenvector is $\begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}$, so $x(t) = e^{(1+i)t} \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}$

is a complex-valued solution of $\dot{x} = Ax$. Now $e^{(1+i)t} = e^t(\cos\theta + i\sin\theta)$, so we can rewrite this solution as

$$\begin{bmatrix} 0 \\ ie^t(\cos\theta + i\sin\theta) \\ e^t(\cos\theta + i\sin\theta) \end{bmatrix} = \begin{bmatrix} 0 \\ -e^t\sin\theta + ie^t\cos\theta \\ e^t\cos\theta + ie^t\sin\theta \end{bmatrix} = e^t\begin{bmatrix} 0 \\ -\sin\theta \\ \cos\theta \end{bmatrix} + ie^t\begin{bmatrix} 0 \\ \cos\theta \\ \sin\theta \end{bmatrix}.$$

By Lemma 4.2, $x^2(t) = e^t\begin{bmatrix} 0 \\ -\sin\theta \\ \cos\theta \end{bmatrix}$ and $x^3(t) = e^t\begin{bmatrix} 0 \\ \cos\theta \\ \sin\theta \end{bmatrix}$. The solutions $x^i(t)$ for $i \in \{1,2,3\}$ are LI since $x^1(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $x^2(0) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $x^3(t) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ which are clearly LI (they form a standard basis for $\mathbb{R}^3$). So a solution must be of the form

$$x(t) = c_1e^t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2e^t\begin{bmatrix} 0 \\ -\sin\theta \\ \cos\theta \end{bmatrix} + c_3e^t\begin{bmatrix} 0 \\ \cos\theta \\ \sin\theta \end{bmatrix}.$$

At $t = 0$, note that $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, so we conclude that

$$x(t) = e^t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + e^t\begin{bmatrix} 0 \\ -\sin\theta \\ \cos\theta \end{bmatrix} + e^t\begin{bmatrix} 0 \\ \cos\theta \\ \sin\theta \end{bmatrix} = e^t\begin{bmatrix} 1 \\ \cos\theta - \sin\theta \\ \cos\theta + \sin\theta \end{bmatrix}.$$

Just pretend that I didn't switch around t and θ for the entire problem.

5.10 Equal Roots

To solve $\dot{x} = Ax$ where $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, note that in the characteristic $(1-\lambda)^2$, the eigenvalue 1 has a multiplicity of two, and the eigenvector corresponding to it is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. So $x^1(t) = e^t\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is one solution. Applying step (2) of the algorithm, we want to find all solutions for $(A-\lambda I)^2v = 0$ with $\lambda = 1$: note that $A - \lambda I$ is a zero divisor when squared, so we conveniently choose $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, since it isn't a multiple of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and hence doesn't satisfy the equation $(A-\lambda I)v = 0$. Therefore,

$$x^2(t) = e^{At}v = e^{\lambda t}[v + t(A-\lambda I)v] = e^t\begin{bmatrix} 1 \\ 2t \end{bmatrix}$$

is a second solution of $\dot{x} = Ax$. So $x(t) = c_1x^1(t) + c_2x^2(t) = c_1e^t\begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2e^t\begin{bmatrix} 1 \\ 2t \end{bmatrix} = \begin{bmatrix} c_2e^t \\ c_1e^t + 2c_2te^t \end{bmatrix}$.

Example 5.7. We want to find three LI solutions for $\dot{x} = Ax$, where $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. The characteristic is $(1-\lambda)^2(2-\lambda)$, so the solutions are $\lambda = 1$ with a multiplicity of two, and $\lambda = 2$. So $x^1(t) = e^t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and

$x^2(t) = e^{2t}\begin{bmatrix}0\\0\\1\end{bmatrix}$ are two LI solutions. Now we want to find all solutions for $(A-\lambda I)^2 v = 0$, let $\lambda = 1$. Then

$$(A-\lambda I)^2 = \begin{bmatrix}(1-\lambda)^2 & 2(1-\lambda) & 0\\ 0 & (1-\lambda)^2 & 0\\ 0 & 0 & (2-\lambda)^2\end{bmatrix} = \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$$

when evaluated. Then for $v = \begin{bmatrix}0\\1\\0\end{bmatrix}$ this is zero but not zero at $(A-\lambda I)v$, since v is LI with the other eigenvectors. Therefore

$$x^3(t) = e^{At}v = e^{\lambda t}\left[v + t(A-\lambda I)v\right] = e^t\begin{bmatrix}t\\1\\0\end{bmatrix}$$

is a third solution for the equation $\dot{x} = Ax$, and so a general solution is of the form

$$c_1 e^t\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2 e^{2t}\begin{bmatrix}0\\0\\1\end{bmatrix} + c_3 e^t\begin{bmatrix}t\\1\\0\end{bmatrix} = \begin{bmatrix}c_1 e^t + c_3 t e^t\\ c_3 e^t\\ c_2 e^{2t}\end{bmatrix}.$$

5.11 Fundamental Matrix Solutions; e^{At}

Example 5.8. To find a fundamental matrix solution for the system

$$\dot{x} = \begin{bmatrix}1 & -1 & 4\\ 3 & 2 & -1\\ 2 & 1 & -1\end{bmatrix}x,$$

since we have

$$e^t\begin{bmatrix}-1\\4\\1\end{bmatrix}, \quad e^{3t}\begin{bmatrix}1\\2\\1\end{bmatrix} \quad \text{and} \quad e^{-2t}\begin{bmatrix}-1\\1\\1\end{bmatrix}$$

three LI solutions, it can be seen that the fundamental matrix is

$$X(t) = \begin{bmatrix}-e^t & e^{3t} & -e^{-2t}\\ 4e^t & 2e^{3t} & e^{-2t}\\ e^t & e^{3t} & e^{-2t}\end{bmatrix}.$$

Example 5.9. We'll show that $X(t) = e^{2t}\begin{bmatrix}4 & 1+4t\\ 2 & 2t\end{bmatrix}$ is a fundamental matrix solution for some A, and we'll also determine A. Now $X(0) = \begin{bmatrix}4 & 1\\ 2 & 0\end{bmatrix}$ and thus has nonzero determinant. If $X(t)$ is a fundamental matrix solution of $\dot{x} = Ax$, then $A = \dot{X}(t)X^{-1}(t)$. We have

$$\dot{X}(t) = \left(e^{2t}\right)'\begin{bmatrix}4 & 1+4t\\ 2 & 2t\end{bmatrix} + e^{2t}\begin{bmatrix}4' & (1+4t)'\\ 2' & (2t)'\end{bmatrix} = 2e^{2t}\begin{bmatrix}4 & 3+4t\\ 2 & 1+2t\end{bmatrix}, \quad X^{-1}(t) = -\frac{1}{2e^{2t}}\begin{bmatrix}2t & -1-4t\\ -2 & 4\end{bmatrix}.$$

So

$$
\begin{aligned}
A &= \dot{X}(t)X^{-1}(t) \\
&= 2e^{2t}\begin{bmatrix} 4 & 3+4t \\ 2 & 1+2t \end{bmatrix} \cdot \left(-\frac{1}{2e^{2t}}\right)\begin{bmatrix} 2t & -1-4t \\ -2 & -4 \end{bmatrix} \\
&= -\begin{bmatrix} 4 & 3+4t \\ 2 & 1+2t \end{bmatrix}\begin{bmatrix} 2t & -1-4t \\ -2 & -4 \end{bmatrix} \\
&= \begin{bmatrix} 6 & -8 \\ 2 & -2 \end{bmatrix}.
\end{aligned}
$$

Lecture 6

Final exam notes

Oh look, I'm actually taking notes beforehand! The topics seemed kind of difficult, so here we are.

6.1 Equilibrium Points

Consider the differential equation $\dot{x} = f(t,x)$, where $x = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$, and $f(t,x) = \begin{bmatrix} f_1(t, x_1, \cdots, x_n) \\ \vdots \\ f_n(t, x_1, \cdots, x_n) \end{bmatrix}$ is a nonlinear function of $x_1, \cdots, x_n$. We have no way to solve these types of differential equations explicitly, but sometimes that's not what we're after: sometimes we just want to know, do there exist ξ_1, ξ_2 such that $x_1(t) = \xi_1$, $x_2(t) = \xi_2$ is a solution of $\dot{x} = f(t,x)$? If ξ_1, ξ_2 exist they're called **equilibrium points** of the differential equation above.

6.2 Stability of Linear Systems

Consider $\dot{x} = f(x)$, where $x = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$ and $f(x) = \begin{bmatrix} f_1(x_1, \cdots, x_n) \\ \vdots \\ f_n(x_1, \cdots, x_n) \end{bmatrix}$. Let $x = \phi(t) = \begin{bmatrix} \phi_1(t) \\ \vdots \\ \phi_n(t) \end{bmatrix}$ be a solution of $\dot{x} = f(t,x)$: we are interested in finding out whether $\phi(t)$ is stable or unstable, that is, we want to find all solutions $\psi(t) = \begin{bmatrix} \psi_1(t) \\ \vdots \\ \psi_n(t) \end{bmatrix}$ where $\psi(t)$ "stays close" to $\phi(t)$ given it "starts near" to $\phi(t)$. Let's make this precise.

Definition 6.1 (Stability). We say a solution $x = \phi(t)$ of $\dot{x} = f(t,x)$ is **stable** if for all $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon)$[2] such that

$$|\psi_j(0) - \phi_j(0)| < \delta \implies |\psi_j(t) - \phi_j(t)| < \varepsilon$$

for all solutions $\psi(t)$, $j \in \mathbb{N}$. You can negate this to define **unstable** solutions (there exists a solution $\psi(t)$ such that for all $\delta > 0$, there exists an $\varepsilon > 0$ such that $|\psi_j(0) - \phi_j(0)| < \delta$ *and* $|\psi_j(t) - \phi_j(t)| \geq \varepsilon$).

Theorem 6.1. *Consider the linear differential equation $\dot{x} = Ax$. Then*

(a) Every solution $x = \phi(t)$ is stable if all the eigenvalues of A have negative real part.

(b) Every solution $x = \phi(t)$ is unstable if at least one eigenvalue of A has positive real part.

(c) Suppose that all eigenvalues of A have real part ≤ 0 and that the eigenvectors $\lambda_1 = ic_1$, $\lambda_2 = ic_2, \cdots \lambda_n = ic_n$ have zero real part. Let $\lambda_i = ic_i$ have multiplicity k_n. Then the characteristic of A can be factored into the form

$$p(\lambda) = (\lambda - ic_1)^{k_1} \cdots (\lambda - ic_n)^{k_n} q(\lambda),$$

where are the roots of $q(\lambda)$ have negative real part. Then every solution $x = \phi(t)$ is stable if A has k_i LI eigenvectors for each eigenvalue $\lambda_i = ic_i$. Otherwise every solution $\phi(t)$ is unstable.

To see why this is true, recall that solutions are of the form $\sum_i c_i e^{\lambda_i t} v_i$ for λ_i eigenvalues, v_i eigenvectors. Then plug in values for λ_i and see whether or not they explode. For part (c), we're just carefully writing out the charateristic, and if any eigenvalue doesn't have the proper amount of LI eigenvectors, you add a t (by judicious guessing) then it blows up.

[2] This notation just means that δ depends on ε.

6.3 The Phase-Plane

Goal, obtain a complete description of all solutions to the equation below. To do this, note that every solution $x = x(t)$, $y = y(t)$ defines a curve in $\mathbb{R}^3$.

$$\frac{dx}{dt} = f(x, y), \quad \frac{dy}{dt} = g(x, y).$$

Key observation is that every solution $x = x(t), y = y(t), t_0 \leq t \leq t_1$ defines a curve in the xy plane, as $t \to t_0$ the set $(x(t), y(t))$ make a curve Γ in the xy plane. Γ is the **orbit** or **trajectory** of the solution and the xy plane is called the **phase-plane** of the solutions. We can get orbits without knowledge of the solution, let $x = x(t), y = y(t)$ be solutions, then if $x'(t) \neq 0$ at $t = t_1$, then we can solve for $t = t(x)$ in a nbd of $x_1 = x(t_1)$. So for t near t_1 the orbit is the curve $y = y(t(x))$, also note that $\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)}$, so orbits of the solutions $x = x(t)$, $y = y(t)$ are just solution curves of the first order ODE $\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)}$. Note that if $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are both zero then the solution curve fails to be an orbit, but rather the union of distinct orbits.

6.4 Phase Portraits of Linear Systems

Printed

6.5 Two Point Boundary-Value Problems

We have a dilemma. For what values of λ can we find nontrivial solutions $y(x)$ satisfying

$$\frac{d^2y}{dx^2} + \lambda y = 0; \quad ay(0) + by'(0) = 0, \quad cy(\ell) + dy'(\ell) = 0? \tag{14}$$

These are called boundary value problems, since we talk about $y(x)$ and $y'(x)$ at two points, $x = 0$ and $x = \ell$. For an IVP, recall that we talk about the value of y at the singular point $x = x_0$. The meat of this section is in the example below.

6.6 Introduction to Partial Differential Equations

Oh look it's the Cauchy-Riemann equations! This section is not important.

6.7 The Heat Equation; Separation of Variables

The meat of this is that solutions to the heat equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial t^2}$, $u(x, 0) = f(x)$, $0 < x < \ell$, $u(0, t) = u(\ell, t) = 0$ are of the form

$$u(x, t) = \sum_{n=1}^{N} c_n \sin\left(\frac{n\pi x}{\ell}\right) e^{-\alpha^2 n^2 \pi^2 t/\ell^2} = \sum_{n=1}^{N} c_n \sin\left(\frac{n\pi x}{\ell}\right) e^{-\alpha^2 \left(\frac{n\pi}{\ell}\right)^2 t}.$$

For separation of variables, split $u(x, t) = X(t)T(t)$, note that they must be equal to a constant and rewrite. To make precise the constant claim, say two functions $f(x)$ and $g(t)$ of x and t respectively are equal. Fix t_0 on the domain, then $f(x) = g(t_0)$, so $f(x) =$ point, therefore so does $g(t)$.

6.8 Fourier Series

Theorem 6.2. *Let f and f' be piecewise continuous on $[-\ell, \ell]$, and compute the values*

$$a_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos\left(\frac{n\pi x}{\ell}\right) dx, \ n \in \mathbb{Z}^+ \cup \{0\}, \quad b_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin\left(\frac{n\pi x}{\ell}\right) dx, \ n \in \mathbb{Z}^+.$$

Then we can form the infinite series

$$\frac{a_0}{2} + a_1 \cos\left(\frac{\pi x}{\ell}\right) + b_1 \sin\left(\frac{\pi x}{\ell}\right) + a_2 \cos\left(\frac{2\pi x}{\ell}\right) + b_2 \sin\left(\frac{2\pi x}{\ell}\right) + \cdots = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{\ell}\right) + b_n \sin\left(\frac{n\pi x}{\ell}\right)\right].$$

This is called the **Fourier series** *for f on $[-\ell,\ell]$, which converges to $f(x)$ if f is continuous at x in the compact interval, and to $\frac{1}{2}[f(x+0)+f(x-0)]$ if f is discontinuous at x, $-\ell < x < \ell$. Here, $f(x+0)$ denotes the limit from the right at x, and similarly $f(x-0)$ the limit from the left. At $x=\pm\ell$, the Fourier series converges to $\frac{1}{2}[f(\ell)+f(-\ell)]$ where $f(\pm\ell)$ is the limit of $f(x)$ as x approaches $\pm\ell$, denoted $\lim_{x\to\ell^-} f(x)$ and $\lim_{x\to-\ell^+} f(x)$ respectively.*

Remark 6.1. Fourier expansions are unique.

6.9 Even and Odd Functions

Even functions look like $f(-x)=f(x)$, **odd functions** look like $f(-x)=-f(x)$, yada yada. Prototypical examples: x^2 is even, x^3 is odd, cos is even, sin is odd. Many function are neither even nor odd. Does this course assume we skipped precalculus or something? Even and odd functions are closed under multiplication (and composition), and odd times even is odd (odd composed with even is even). Also the obvious fact about integrals of even and odd functions (watch MIT's integration bee).

Lemma 6.1. *The Fourier expansion for an even function is a pure cosine series, and similarly the Fourier expansion of an odd function is all sine.*

Theorem 6.3. *Let f and f' be piecewise continuous on the interval $[0,\ell]$, and compute*

$$a_n = \frac{2}{\ell}\int_0^\ell f(x)\cos\left(\frac{n\pi x}{\ell}\right)dx,\ n\in\mathbb{Z}^+\cup\{0\},\quad b_n = \frac{2}{\ell}\int_0^\ell f(x)\sin\left(\frac{n\pi x}{\ell}\right)dx,\ n\in\mathbb{Z}^+.$$

We can then form the infinite series

$$\frac{a_0}{2}+\sum_{n=1}^\infty a_n\cos\left(\frac{n\pi x}{\ell}\right),\quad \sum_{n=1}^\infty b_n\left(\frac{n\pi x}{\ell}\right),$$

both series converge to $f(x)$ if f is continuous at $x\in(0,\ell)$, and to $\frac{1}{2}[f(x+0)+f(x-0)]$ if f is discontinuous at $x\in(0,\ell)$. At $x=0$ and $x=\ell$ the first series converges to $f(x)$ and the second series to 0.

6.10 Return to the Heat Equation

Back to the boundary value problem

$$\frac{/\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2};\quad u(x,0)=f(x),\ 0<x<\ell;\quad u(0,t)=u(\ell,t)=0.$$

We've shown (not really, skipped over it but trust me) that $u(x,t)=\sum_{n=1}^\infty c_n\sin\left(\frac{n\pi x}{\ell}\right)e^{-\alpha^2\left(\frac{n\pi}{\ell}\right)^2 t}$. This leads us to ask whether we can find constants c_1,c_2 such that $u(x,0)=\sum_{n=1}^\infty c_n\sin\left(\frac{n\pi x}{\ell}\right)=f(x), x\in[0,\ell]$. The answer is yes as we have seen, if we choose $c_n=\frac{2}{\ell}\int_0^\ell f(x)\sin\left(\frac{n\pi x}{\ell}\right)dx$, then the Fourier series $\sum_{n=1}^\infty c_n\sin\left(\frac{n\pi x}{\ell}\right)$ converges to $f(x)$ if f is continuous at x. Therefore the solution we want is given by

$$u(x,t)=\frac{2}{\ell}\sum_{n=1}^\infty\left[\int_0^\ell f(x)\sin\left(\frac{n\pi x}{\ell}\right)dx\right]\sin\left(\frac{n\pi x}{\ell}\right)e^{-\alpha^2\left(\frac{n\pi}{\ell}\right)^2 t}$$

Lecture 7

Examples

7.1 Equilibrium Points

Example 7.1. To find all equilibrium values of the system of differential equations

$$\frac{dx_1}{dt} = 1 - x_2,\ \frac{dx_2}{xt} = x_1^3 + x_2,$$

note that $x^0 = \begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix}$ is an equilibrium point iff $1 - x_2^0 = 0$ and $(x_1^0)^3 + x_2^0 = 0$, so $x_2^0 = 1$ and $x_1^0 = -1$, so $\begin{bmatrix} -1 \end{bmatrix}$ is the only equilibrium value of this system.

Example 7.2. To find all equilibrium values of

$$\frac{dx}{dt} = (x-1)(y-1),\ \frac{dy}{dt} = (x+1)(y+1),$$

consider $x^0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ an equilibrium value iff $(x_0 - 1)(y_0 - 1) = 0$ and $(x_0 + 1)(y_0 + 1) = 0$. The first equation holds if either x_0 or y_0 is 1, while the second holds if either x_0 or y_0 is -1. So the equilibrium values are $x = 1, y = -1$ and $x = -1, y = 1$.

7.2 Stability of Linear Systems

Example 7.3. To show every solution of $\dot{x} = Ax$ where $A = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$ is unstable, note that the characteristic is $(1-\lambda)^2 - 25$ so the eigenvalues are $\lambda = 6, -4$, so we have a positive eigenvalue and we are done.

Example 7.4. To show every solution where $A = \begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix}$ is stable, note that the characteristic is $(\lambda - \sqrt{6}i)(\lambda + \sqrt{6}i)$, so the eigenvalues are $\lambda = \pm\sqrt{6}i$, and the eigenvectors add up, thus the solutions are stable.

Example 7.5. Now let $A = \begin{bmatrix} 2 & -3 & 0 \\ 0 & -6 & -2 \\ -6 & 0 & -3 \end{bmatrix}$. The characteristic is $-\lambda^2(7+\lambda)$, and the eigenvector corresponding to $\lambda = 0$ is $\begin{bmatrix} 3 \\ 2 \\ -6 \end{bmatrix}$, so every solution is unstable since the eigenvectors don't add up.

7.3 The Phase-Plane

Example 7.6. The solution $x = \cos t, y = \sin t$ of the system of differential equations $\frac{dx}{dt} = -y$, $\frac{dy}{dt} = x$ describes a helix in (t, x, y) space. Also, the solutions $x = e^{-t}\cos t,\ y = e^{-t}\sin t$ for $-\infty < t < \infty$ of the system $\frac{dx}{dt} = -x - y$, $\frac{dy}{dt} = x - y$ trace out a spiral in the xy plane.

Example 7.7. The orbits of the system $\frac{dx}{dt} = y^2$, $\frac{dy}{dt} = x^2$ are solution curves of $\frac{dy}{dx} = \frac{x^2}{y^2}$ which are clearly of the form $y(x) = (x^3 + C)^{1/3}$ for $C \in \mathbb{R}$. Similarly, the orbits of $\frac{dx}{dt} = y(1 + x^2 + y^2)$, $\frac{dy}{dt} = -2x(1 + x^2 + y^2)$ are solution curves of $\frac{dy}{dx} = -\frac{2x}{y}$ which is the family of ellipses $\frac{1}{2}y^2 + x^2 = C^2$.

7.4 Phase Portraits of Linear Systems

Just follow the examples

7.5 Two Point Boundary-Value Problems

Example 7.8. For what values of λ does Equation (14) have nontrivial solutions for $a = 1, b = 0, c = 1, d = 0$? $(y(0) = 0, y(\ell) = 0)$. We talk about three cases.

(i) $\lambda = 0$. To find the general solution of $y'' = 0$, note that $y' = c_1$ and so $y = c_1 x + c_2$. Since $y(0) = 0$, we have $c_2 = 0$, and $y(\ell) = 0$ implies $c_1 \ell = 0$, thus $c_1 = 0$ since ℓ is nonzero, and we're working in an integral domain. Therefore $y(x) = 0$ is the only solution for Equation (14) given $\lambda = 0$.

(ii) $\lambda < 0$. The characteristic for $y'' + \lambda y = 0$ is $r^2 + \lambda$, and $r = \pm\sqrt{-\lambda}$. If $\lambda < 0$, then $\sqrt{-\lambda}$ is real, and two LI solutions are $y_1(x) = e^{\sqrt{-\lambda}x}$, $y_2(x) = e^{-\sqrt{-\lambda}x}$. Now $y(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$, $y(0) = 0$ and $y(\ell) = 0$ imply that

$$\begin{cases} c_1 + c_2 = 0 \\ c_1 e^{\sqrt{-\lambda}} + c_2 e^{-\sqrt{-\lambda}} \ell = 0 \end{cases}$$

This system has a nonzero solution iff $\begin{vmatrix} 1 & 1 \\ e^{\sqrt{-\lambda}t} & e^{-\sqrt{-\lambda}t} \end{vmatrix} = e^{\sqrt{-\lambda}t} - e^{-\sqrt{-\lambda}t} = 0$, which implies that $e^{\sqrt{-\lambda}t} = e^{-\sqrt{-\lambda}t} \implies e^{2\sqrt{-\lambda}t} = 1$, which is impossible since $e^z > 1$ for $z > 0$. So $c_1 = c_2 = 0$, and thus Equation (14) has no nontrivial solutions when λ is negative.

(iii) $\lambda > 0$. If $\lambda > 0$, then $\sqrt{-\lambda}$ is complex, and since $\sqrt{-\lambda} = 0 + \sqrt{\lambda}i$, two LI solutions are given by $y_1(x) = e^{0x} \cos\sqrt{\lambda}x = \cos\sqrt{\lambda x}$ and $y_2(x) = \sin\sqrt{\lambda}x$, thus the general solution is of the form $y(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$. Now $y(0) = 0$ implies that $c_1 = 0$, and $y(\ell) = 0$ implies that $\sin\sqrt{\lambda}\ell = 0$ since otherwise we would have the trivial solution. This is true (for any c_2) when $\sqrt{\lambda}\ell = n\pi$ for some $n \in \mathbb{N}$, so $\lambda = \frac{n^2\pi^2}{\ell^2}$. So the boundary value problem given by Equation (14) has nontrivial solutions

$$y(x) = c\sin\sqrt{\lambda}x = c\sin\left(\frac{n\pi x}{\ell}\right)$$

for $\lambda = \frac{n^2\pi^2}{\ell^2}$, $n \in \mathbb{N}$.

7.6 Introduction to Partial Differential Equations

Not needed

7.7 The Heat Equation; Separation of Variables

Example 7.9. The random guy from cornell did a much better job with examples of separation of variables, so I'm not even gonna bother. It's in the quizzes.

7.8 Fourier Series

Example 7.10. Let

$$f(x) = \begin{cases} 0 & \text{for } -1 \le x < 0 \\ 1 & \text{for } 0 \le x \le 1. \end{cases}$$

To compute the Fourier expansion of f on $[-1, 1]$, note that

$$a_0 = \frac{1}{1}\int_{-1}^{1} f(x)\cos\left(\frac{0\pi x}{1}\right)dx = \int_{-1}^{1} f(x)\,dx = 1.$$

For $n \ge 1$, we have

$$a_n = \frac{1}{1}\int_{-1}^{1} f(x)\cos\left(\frac{n\pi x}{1}\right)dx = \int_{-1}^{0} f(x)\cos(n\pi x)\,dx + \int_{0}^{1} f(x)\cos(n\pi x)\,dx = 0 + \frac{1}{n\pi}\sin(n\pi x)\bigg|_0^1 = 0,$$

and similarly

$$b_n = \int_{-1}^{0} 0\cdot\sin(n\pi x)\,dx + \int_0^1 1\cdot f(x)\sin(n\pi x)\,dx = -\frac{1}{n\pi}\cos(n\pi x)\bigg|_0^1 = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \frac{2}{n\pi} & \text{if } n \text{ is odd.}\end{cases}$$

So the Fourier expansion is

$$\frac{1}{2} + \frac{2}{\pi}\sin(\pi x) + \frac{2}{3\pi}\sin(3\pi x) + \cdots = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)\pi x}{2n-1}.$$

By the big thm this converges to 0 if $-1 < x < 0$ and 1 if $0 < x < 1$. At $x = 0, \pm 1$ this series reduces to $1/2$, just plug it in, this was also foretold by the thm. To elaborate, $\frac{1}{2}[f(x+0)+f(x-0)] = \frac{1}{2}[1+0] = \frac{1}{2}$, and $\frac{1}{2}[f(1)+f(-1)] = \frac{1}{2}[1+0] = \frac{1}{2}$ at the endpoints.

Example 7.11. Let

$$f(x) = \begin{cases} 1 & \text{for } -2 \le x < 0 \\ x & \text{for } 0 \le x \le 2. \end{cases}$$

We compute the Fourier expansion of f on $[-2, 2]$. Note that $a_0 = 2$, and

$$a_n = \frac{1}{2}\int_{-2}^{0}\cos\left(\frac{n\pi x}{2}\right)dx + \frac{1}{2}\int_0^2 x\cos\left(\frac{n\pi x}{2}\right)dx = \frac{2}{n^2\pi^2}(\cos(n\pi)-1) \quad \text{for } n \ge 1,$$
$$b_n = \frac{1}{2}\int_{-2}^{0}\sin\left(\frac{n\pi x}{2}\right)dx + \frac{1}{2}\int_0^2 x\sin\left(\frac{n\pi x}{2}\right)dx = -\frac{1}{n\pi}(1+\cos(n\pi)) \quad \text{for } n \ge 1.$$

Note that $a_n = 0$ if n is even, and $-\frac{4}{n^2\pi^2}$ if n is odd ($\cos(n\pi) = (-1)^n$), $b_n = -\frac{2}{n\pi}$ if n is even, and 0 if n is odd. So the Fourier series is given by

$$1 + a_1\cos\left(\frac{\pi x}{2}\right) + b_2\sin\left(\frac{2\pi x}{2}\right) + a_3\cos\left(\frac{3\pi x}{2}\right) + \cdots = 1 - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos\left(\frac{(2n-1)\pi x}{2}\right)}{(2n-1)^2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin(n\pi x)}{n}.$$

Since this series converges to $f(x)$ if f is continuous at x, it converges to 1 if $x \in (-2,0)$, and x if $x \in (0,2)$. At $x = 0$ this becomes $1/2$, and $3/2$ at the endpoints. If we plug in $x = 0$, we get the remarkable identity $\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$.

Example 7.12. Let's find the Fourier series for $f(x) = \cos^2 x$ on $[-\pi, \pi]$. We have a unique Fourier expansion of the form $\frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos(nx) + b_n\sin(nx)]$ on the interval, but we already know by reduction that

$$\cos^2 x = \frac{1+\cos(2x)}{2} = \frac{1}{2} + 0\cdot\cos x + 0\cdot\sin x + \frac{1}{2}\cos(2x) + 0\cdot\sin(2x) + \cdots$$

So this must be it. This shows the power of existence uniqueness.

7.9 Even and Odd Functions

Example 7.13. This example serves to demonstrate Theorem 6.3. We want to expand $f(x) = 1$ on the interval $[0, \pi]$. Expanding f in a pure sine series on $(0, \pi)$, we have $f(x) = \sum_{n=1}^{\infty} b_n(nx)$, where

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \frac{4}{n\pi} & \text{if } n \text{ is odd.}\end{cases}$$

So

$$f(x) = \sum_{k=1}^{\infty} b_{\text{odd}}\sin(2k-1) + \sum_{k=1}^{\infty} b_{\text{even}}\sin(2k)x = \sum_{k=1}^{\infty}\frac{4}{(2k-1)\pi}\sin(2k-1)x = \frac{4}{\pi}\left(\sin x + \frac{\sin(3x)}{3} + \cdots\right)$$

Then $\sin x + \frac{\sin(3x)}{3} + \cdots = \frac{\pi}{4}$ for any $x \in (0,\pi)$. Choose $x = \frac{\pi}{2}$, then we get the wonderful fact that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} = \frac{\pi}{4}$$

Now expanding f in a pure cosine series gets $a_0 = 2$, $a_n = 0$ for $n \in \mathbb{N}$. So $\frac{2}{2} + \sum_{n=1}^{\infty} 0 \cos\left(\frac{n\pi x}{\ell}\right) = 1$. Theorem 6.3 gives the same result, since

$$a_0 = \frac{2}{\pi}\int_0^{\pi} 1\, dx = 2, \quad a_n = \frac{2}{\pi}\int_0^{\pi} 1 \cos(nx)\, dx = \frac{2}{n\pi}(0-0) = 0.$$

Example 7.14. Let's expand $f(x) = e^x$ into a pure cosine series on $[0,1]$. We have $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) =$, where

$$a_0 = \frac{2}{1}\int_0^1 e^x\, dx = 2(e-1), \quad a_n = 2\int_0^1 e^x \cos(n\pi x)\, dx = \frac{2(e\cos(n\pi)-1)}{1+n^2\pi^2}.$$

We can solve the second integral by parts or writing $\cos(n\pi x) = \operatorname{Re} e^{in\pi x}$ and taking real parts of the solution.

Example 7.15. Expand the following function into a pure cosine series on $[0,2]$.

$$f(x) = \begin{cases} 0 & \text{for } 0 \le x \le 1 \\ 1 & \text{for } 1 < x \le 2. \end{cases}$$

By Theorem 6.3 we have $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{1}\right)$, where $a_0 = \frac{2}{2}\int_0^2 f(x)\, dx = 1$, and

$$a_n = \int_0^2 f(x)\cos\left(\frac{n\pi x}{2}\right) dx = \int_1^2 \cos\left(\frac{n\pi x}{2}\right) dx = \frac{2}{n\pi}\left(0 - \sin\left(\frac{n\pi}{2}\right)\right) = \begin{cases} 0 & \text{if } n \text{ is even} \\ -\frac{2}{n\pi} & \text{if } \frac{n-1}{2} \text{ is even} \\ \frac{2}{n\pi} & \text{if } \frac{n-1}{2} \text{ is odd.} \end{cases}$$

So

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^k}{2k-1} \cos\left(\frac{(2k-1)\pi x}{2}\right),$$

given $x \in [0,2] \setminus \{1\}$. At $x = 1$, this reduces to $1/2$, the expected value since

$$\frac{1}{2}\left[\lim_{x\to 1^+} f(x) + \lim_{x\to 1^-} f(x)\right] = \frac{1}{2}[1+0] = \frac{1}{2}.$$

7.10 Return to the Heat Equation

Problem. *A thin aluminum bar ($\alpha^2 = 0.86$ cm^2 /s) 10 cm long is heated to a uniform temperature of 100°C. At time $t = 0$, the ends of the bar are plunged into an ice bath at 0°C, and thereafter they are maintained at this temperature. No heat is allowed to escape through the lateral surface of the bar. Find an expression for the temperature at any point in the bar at any later time t.*

Solution. Let $u(x,t)$ denote the temperature in the bar at the point x at time t. This satisfies the boundary value problem

$$\frac{\partial u}{\partial t} = 0.86\frac{\partial^2 u}{\partial x^2}; \quad u(x,0) = 100,\ 0 < x < 10; \quad u(0,t) = u(10,t) = 0.$$

The solution is therefore $u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{10}\right) e^{-0.86 n^2\pi^2 t/100}$, where

$$c_n = \frac{1}{5}\int_0^{10} 100 \sin\left(\frac{n\pi x}{10}\right) dx = \frac{200}{n\pi}(1-\cos(n\pi)) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \frac{400}{n\pi} & \text{if } n \text{ is odd.} \end{cases}$$

So

$$u(x,t) = \sum_{k=1}^{\infty} c_{\text{odd}} \sin\left(\frac{(2k-1)\pi x}{10}\right) e^{-0.86(2k-1)^2\pi^2 t/100} + \sum_{k=1}^{\infty} c_{\text{even}} \text{blah} = \frac{400}{\pi}\sum_{k=1}^{\infty} \frac{\sin\left(\frac{(2k-1)\pi x}{10}\right)}{2k-1} e^{-0.0086((2k-1)\pi)^2 t}.$$

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