Algebraic Geometry Notes

Simon Xiang

January 22, 2022

Notes for the Fall 2020 graduate section of Algebraic Geometry (Math 392C) at UT Austin, taught by Dr. Ben-Zvi. The course follows *The Rising Sea*, by Ravi Vakil. Source files: https://git.simonxiang.xyz/math_notes/files.html

Contents

1 August 26, 2021

Algebraic geometry is about trying to think about algebraic objects in terms of geometry, for example visualizing $x^2 + y^2 = 1$ as the circle. For example, we can discuss things like symmetries, groups, parametrizations, calculus, length (2π), the topology, etc. Grothendieck says that we should be able to do this to *all* algebraic objects.

There is a natural map from Geometry $\to$ Algebra by looking at functions. Let k be a field, like $\mathbb{R}$ or $\mathbb{C}$. Let X be a set or space that we do geometry. Then consider the map

$$X \mapsto \mathrm{Fun}_k(X) := \{f : X \to k\},$$

where $\mathrm{Fun}_k(X)$ forms a commutative ring. Addition is given by $f(x) + g(x) = (f + g)(x)$ and multiplication is $f(x) \cdot g(x) = (fg)(x)$. For example, if $X = S^1 \subseteq \mathbb{R}^2$, consider polynomial functions on X, the restriction $\{\text{polynomials on } \mathbb{R}^2\} = k[x, y]$. This ring of polynomial functions $\mathrm{Poly}(S^1) \simeq k[x, y]/(x^2 + y^2 - 1)$ naturally recovers our polynomials valued on the circle.

The first topic we define is the notion of a **scheme**, a class of geometric spaces or setting to do geometry. Inside of schemes we have $\operatorname{Spec} R$, where R is a commutative ring.

1.1 Preamble to algebraic geometry

Let $R = \mathbb{Z}[a, b, c]/(a^n + b^n - c^n)$, which results in a geometric object called $\operatorname{Spec} R$ (analogously with $\operatorname{Spec} \mathbb{Z}$). "Spectrum" refers to a rainbow, like the spectrum of an operator, a subset of $\mathbb{R}$ given by eigenvalues on a matrix. If $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$, we look at $\mathbb{R}[x]/(\text{characterisitic polynomials of } A)$. Algebraic topologists also like the work spectrum, but these ideas are unrelated.

Let X be a space, and $x \in X$. How do points relate to functions? Given a function, we can evaluate it at that point. Let $R = \mathrm{Fun}_k(X)$, then we get a ring homomorphism $\mathrm{eval}_x : R \to k$, where $f \mapsto f(x)$. Then $\ker(\mathrm{eval}_x) = m_x = \{f / f(x) = 0\} \subseteq R$, which is a *maximal ideal*. Modding out by maximal ideals gives a field, and the kernel is maximal since if $m_x \subseteq I \subseteq R$, $0 \subseteq I/m \subseteq R/m = k$. This results in an ideal between 0 and a field, therefore m_x is maximal.

The basic algorithm is that points $= R \rightsquigarrow$ maximal ideals of R ($R/m \simeq k$), which we want to realize as $R = \mathrm{Fun}_k(X)$. Gelfand duality says that if X is a compact Hausdorff topological space, then $X \mapsto C(X)$ where $C(X)$ is a commutative C^*-algebra. So this idea is not unique to algebraic geometry. We want to take this idea and modify it to work for arbitrary commutative rings. In particular, we allow

- Rings with nilpotent elements, eg $\varepsilon^2 = 0$. This was outlawed in classical AG. In the world of schemes, this corresponds to analysis, in particular derivatives.
- We are not going to assumed that our fields are algebraically closed, ($k = \overline{k}$), or even a field at all. This corresponds to arithemtic, since one of the things we want to do is study the integers by AG.
- The third generalization of classical AG is gluing. Just one $\operatorname{Spec} R$ is not all the spaces we want, we glue them together like manifolds which corresponds to topology.

These are the main differences between pre-Grothendieck AG and modern AG.

1.2 Introduction to category theory

Category theory is an indispensible tool in many areas of math. There isn't an obvious home for it in the curriculum, so this is the home.

Definition 1.1. A category $\mathcal{C}$ consists of two collections[sets][1]:

- $\mathrm{Ob}\,\mathcal{C}$, the set of objects,
- $\mathrm{Mor}\,\mathcal{C} \to \mathrm{Ob}\,\mathcal{C} \times \mathrm{Ob}\,\mathcal{C}$, the set of morphisms.

For $X, Y \in \mathrm{Ob}\,\mathcal{C}$, this maps to $\mathrm{Hom}_{\mathcal{C}}$ todo:go back and finish. Finally, we have a composition: for all $X, Y, Z \in \mathcal{C}$,

$$\overbrace{\mathrm{Hom}_{\mathcal{C}}(X,Y)}^{f} \times \overbrace{\mathrm{Hom}_{\mathcal{C}}(X,Y)}^{g} \to \overbrace{\mathrm{Hom}_{\mathcal{C}}(X,Y)}^{g\circ f}$$

todo:finish def

Example 1.1. A category $\mathcal{C}$ with one object X has the data $\mathrm{Hom}_{\mathcal{C}}(X,X) := A$. Then A is a monoid, since $1_X \in A$ is the unit and multiplication is composition. If we require the morphism to be invertible, then A is a group.

Definition 1.2. We say $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ is an **isomorphism** if there exists a $g \in \mathrm{Hom}_{\mathcal{C}}(Y,X)$ such that $f \circ g = 1_Y$, $g \circ f = 1_X$. A category $\mathcal{C}$ is a **groupoid** if all morphisms are isomorphisms.

Example 1.2. A groupoid with only one object is a group. The most famous groupoid is the fundamental groupoid (or Poincaré groupoid) $\pi_{\leq 1}(X)$, where $\mathrm{Ob} := \{x \in X\}$ points, and morphisms are $\mathrm{Hom}(x,y) =$ paths from x to y up to homotopy. In other words, this is $\pi_1(X)$ without reliance on the basepoint.

Example 1.3 (Posets as categories)**.** Let S be a set with a partial order $\leq$, that is for all $x, y, z \in X$,

- $x \leq x$,
- $x \leq y \leq z \implies x \leq z$,
- $x \leq y$ and $y \leq x \implies x = y$.

This gives rise to a category $\mathcal{C}$, where $\mathrm{Ob}\,\mathcal{C} = S$, and morphisms are $\mathrm{Hom}_{\mathcal{C}}(X,Y) = \{x \leq y\}$ (there is an arrow from x to y if $x \leq y$). This is a **small** category.

Example 1.4. Here are examples of "big" categories. Before, we could list the objects.

- Set is the category where objects are sets and morphisms are maps between sets.
- Group is the category where objects are groups and morphisms are groups homomorphisms.
- Top is the category of topological spaces with continuous maps as morphisms.
- Ring is the category of rings with ring homomorphisms.
- R-Mod is the category of modules over a ring R, where k-modules are just vector spaces in Vect_k and $\mathbb{Z}$-modules are just abelian groups in Ab.

Objects are irrelevant. The whole point of category theory is we should never think about categories without thinking about the morphisms. This leads into the next idea: we have categories as objects, but how do two categories talk to each other?

Definition 1.3. Let $\mathcal{C}, \mathcal{D}$ be categories. Then a **functor** $F : \mathcal{C} \to \mathcal{D}$ is

- a function $\mathrm{Ob}\,\mathcal{C} \xrightarrow{F} \mathrm{Ob}\,\mathcal{D}$, where $X \in \mathcal{C} \mapsto F(X) \in \mathcal{D}$,

[1] We can't say sets because of some logic issue with Russell's paradox, but think of them as sets.

- given $X, Y \in \mathcal{C}$,
$$\mathrm{Hom}_{\mathcal{C}}(X,Y) \xrightarrow{F} \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$$
which preserves identities and composition:
$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(Y,Z) \arrow[r] \arrow[d, "F\times F"] & \mathrm{Hom}_{\mathcal{C}}(X,Z) \arrow[d, "F"] \\ \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y)) \times \mathrm{Hom}_{\mathcal{D}}(F(Y),F(Z)) \arrow[r] & \mathrm{Hom}_{\mathcal{D}}(F(X),F(Z)) \end{tikzcd}$$
This diagram *commutes*, eg it doesn't matter which way you compose the arrows.

This defines what we call a **covariant functor**. A **contravariant functor** reverses the direction of arrows, eg $\mathrm{Hom}_{\mathcal{C}}(X,Y) \xrightarrow{F} \mathrm{Hom}_{\mathcal{C}}(F(Y),F(X))$.

In terms of monoids, we can define the opposite monoid as $a \cdot b = b \cdot a$, and we can do the same thing in terms of categories: morphisms go the other way. Therefore we can think of contravariant functors as standard covariant functors in $\mathcal{C}^{\mathrm{op}}$.

Example 1.5 (Functions)**.** This is the example our class is built on. Let $\mathrm{Fun}_k \colon \mathsf{Set} \to \mathsf{Ring}$[2] be given by $X \mapsto \mathrm{Fun}_k(X)$. What make this a functor is that it's compatible with morphisms: given a map $X \to Y$, how do we relate functions on Y to functions on X? Functions pull back:
$$\begin{tikzcd} X \arrow[rr, "\pi"] \arrow[dr, "\pi^* f"'] & & Y \arrow[dl, "f"] \\ & k & \end{tikzcd}$$
This tells us that $(\pi^* f)(x) = f(\pi(x))$. So $\mathrm{Fun}_k(Y) \xrightarrow{\pi^*} \mathrm{Fun}_k(X)$ is a contravariant functor, which we call Spec. This gives rise to the slogan for this class: Algebra=(Geometry)$^{\mathrm{op}}$.

2 August 31, 2021

We continue with our tour of category theory, in the category (commutative) Rings where objects are rings and morphisms are homomorphisms. Eventually we will talk about the category of affine schemes, where the opposite category is Ring.

2.1 Functors

Let Top be the category of topological spaces with continuous maps. Consider the contravariant functor $C \colon \mathsf{Top} \to \mathsf{Vect}_{\mathbb{R}}^{\mathrm{op}}$, where $\mathsf{Vect}_{\mathbb{R}}$ is the category of vector spaces with linear maps. This functor sends $X \mapsto C(X)$, the vector space of $\mathbb{R}$-valued functions on X. If we have $X \xrightarrow{\varphi} Y$ a continuous map, then a real valued function on Y is a map $Y \xrightarrow{f} \mathbb{R}$.
$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "f\circ\varphi"'] & & Y \arrow[dl, "f"] \\ & \mathbb{R} & \end{tikzcd}$$

[2]From now on in this class, all rings are commutative.

So φ gets send to the pullback $C(Y) \xrightarrow{\varphi^* f = f\circ\varphi} C(X)$. Now given a composition $X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z$, for C to satisfy functorality we need the following diagram to commute:

$$\begin{tikzcd} \mathrm{Hom}(X,Y)\times\mathrm{Hom}(Y,Z) \arrow[r] \arrow[d] & \mathrm{Hom}(X,Z) \arrow[d] \\ \mathrm{Hom}(C(X),C(Y))\times\mathrm{Hom}(C(Y),C(Z)) \arrow[r] & \mathrm{Hom}(C(X),C(Z)) \end{tikzcd}$$

In other words, it doesn't matter if we pull back by φ then ψ or we pull back by $(\psi\circ\varphi)$.

There are two ways to compare categories, a useless and a useful way. The useless notion is the category of categories Cat, where objects are categories and morphisms are functors. So given categories $\mathcal{C},\mathcal{D}$, elements of $\mathrm{Hom}_{\mathsf{Cat}}(\mathcal{C},\mathcal{D})$ are functors $F\colon\mathcal{C}\to\mathcal{D}$, and composition is just given by

$$\underset{X}{\mathcal{C}} \xrightarrow{F} \underset{F(X)}{\mathcal{D}} \xrightarrow{G} \underset{G(F(X))}{\mathcal{E}}.$$

We want to define the idea of an *isomorphism of categories*. A functor $F\colon\mathcal{C}\to\mathcal{D}$ is an **isomorphism** if there exists some $G\colon\mathcal{D}\to\mathcal{C}$ such that $F\circ G = \mathrm{id}_{\mathcal{C}}$ and $G\circ F = \mathrm{id}_{\mathcal{D}}$. On sets of objects, $\mathrm{Ob}(\mathcal{C})\simeq\mathrm{Ob}(\mathcal{D})$ as sets. This is a completely useless notion and we will never bring it up again. A better notion is that we shouldn't as for $F\circ G$ and $G\circ F$ to *equal* identities, but to be equivalent to them. This leads into the idea of *natural transformations*.

2.2 Natural transformations and equivalence of categories

The goal is to compare two functors. Suppose we have two functors $F,G\colon\mathcal{C}\to\mathcal{D}$, and we want to compare them, usually with a down arrow like below.

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathcal{D} \end{tikzcd}$$

Definition 2.1. A **natural transformation** is data $X\in\mathcal{C}$ where $\eta_X\colon F(X)\to G(X)$, $\eta_X\in\mathrm{Hom}_{\mathcal{D}}(F(X),G(X))$, such that for $X\in\mathcal{C}, Y\in\mathcal{D}$, the *naturality diagram* commutes:

$$\begin{tikzcd} F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\eta_Y"] & G(Y) \end{tikzcd}$$

Natural isomorphisms are natural transformations where η_X,η_Y are isomorphisms.

With our added information, a better way to think about isomorphisms of categories is by considering the **functor category** $\mathrm{Fun}(\mathcal{C},\mathcal{D})$, where objects are functors $F\colon\mathcal{D}\to\mathcal{D}$ and morphisms are natural transformations $F\xrightarrow{\eta}G$.

Definition 2.2. A functor $F\colon\mathcal{C}\to\mathcal{D}$ is an **equivalence of categories** if there exists a functor (quasi-inverse) $G\colon\mathcal{D}\to\mathcal{C}$ and natural isomorphisms $F\circ G\xrightarrow{\eta}\mathrm{id}_{\mathcal{D}}, G\circ F\xrightarrow{\eta'}\mathrm{id}_{\mathcal{C}}$.

Example 2.1. Let $\mathcal{C}$ be the category with a single object and $\mathrm{Hom}_{\mathcal{C}}(*,*)=\mathrm{id}$. Let $\mathcal{D}$ be a big category such that for all $x,y\in D$, there exists a unique $X\to Y$. Define a functor $F\colon\mathcal{C}\to\mathcal{D}$ by letting $*\mapsto X\in\mathcal{D}$, $\mathrm{id}_*\mapsto\mathrm{id}_X$ by functorality. The claim is that this functor is an equivalence of categories.

To construct a quasi-inverse $G\colon\mathcal{D}\to\mathcal{D}$, we have to send $Y\in\mathcal{D}\mapsto *\in\mathcal{C}$. For morphisms, there is only one morphism $Y\to Z$ between objects Y and Z. So for a map $G(Y)\to G(Z)$, this is just $\mathrm{id} := G(f)$. It remains an exercise to check that this preserves composition. To check that this is indeed a quasi-inverse, the easy direction is checking $G \circ F = \mathrm{id}_{\mathcal{C}}$ (in this case literal equality). The hard direction is checking $F \circ G$ sending Y to X through $* \in \mathcal{C}$, but there is a unique isomorphism $Y \to X$.

In a sense, considering equivalence of categories allows us to "erase" giant trivial categories like $\mathcal{D}$.

Example 2.2. Consider the category of real finite dimensional vector spaces $\mathsf{Vect}_{\mathbb{R}}^{\mathrm{fd}}$ with morphisms just linear maps. Also consider the Euclidian category Euc, where objects $\langle n \rangle$ are non-negative integers $n \in \mathbb{Z}_{\geq 0}$ and morphisms are defined by $\mathrm{Hom}_{\mathsf{Euc}}(\langle n \rangle, \langle m \rangle) = \mathrm{Mat}_{n \times m}$ under matrix multiplication. This is the same as a map $\mathbb{R}^n \to \mathbb{R}^m$, and given a map $\mathbb{R}^m \to \mathbb{R}^k$ we can compose to get a map/matrix $\mathbb{R}^n \to \mathbb{R}^k$.

The claim is that $\mathsf{Vect}_{\mathbb{R}}^{\mathrm{fd}} \simeq \mathsf{Euc}$ are equivalent. The obvious functor is "inclusion" where $F : \mathsf{Euc} \to \mathsf{Vect}_{\mathbb{R}}^{\mathrm{fd}}$, $\langle n \rangle \mapsto \mathbb{R}^n$. This tells us that matrices can be thought of as linear maps, a fact we all know from linear algebra. To construct the inverse, we need to construct a functor $G \colon \mathsf{Vect}_{\mathbb{R}}^{\mathrm{fd}} \to \mathsf{Euc}$. Let $G \colon V \mapsto \langle \dim V \rangle$: we need to send $f \in \mathrm{Hom}(V, W) \mapsto$ a matrix. Choose a basis for V and W, then follow the arrows from $\mathbb{R}^{\dim V} \to \mathbb{R}^{\dim W}$ in the following diagram:

$$\begin{tikzcd}
V \arrow[r, "f"] \arrow[d] & W \arrow[d] \\
\mathbb{R}^{\dim V} \arrow[r, "\text{matrix of } f"] & \mathbb{R}^{\dim W}
\end{tikzcd}$$

The idea is that equivalence of categories does violence to the set of objects, but preserves the essential properties of the category.

Definition 2.3. A functor $F : \mathcal{D} \to \mathcal{D}$ is **faithful** if $F \colon \mathrm{Hom}_{\mathcal{C}}(X, Y) \hookrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ is injective. We say F is **fully faithful** if $\mathrm{Hom}_{\mathcal{C}}(X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ is one-to-one and onto. We also say F is **essentially surjective** if every $Z \in D$ is isomorphic to F(X) for some X.

Theorem 2.1. *A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories iff F is fully faithful and essentially surjective.*

Going back to Example 2.2, the functor $\mathsf{Euc} \to \mathsf{Vect}_{\mathbb{R}}^{\mathrm{fd}}$ is fully faithful with $\langle n \rangle \mapsto \mathbb{R}^n$, $\mathrm{Hom}(\langle n \rangle, \langle m \rangle)$ is the set of $n \times m$ matrices is $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$. Theorem 2.1 allows us to avoid the horrible construction of an inverse by choosing bases and just check properties of the functor itself.

What is preserved by equivalence of categories?

- The set of *isomorphism classes* of objects
- Automorphisms of an object

These two properties allow us to classify groupoids. For example, consider the category of finite sets up to isomorphism $\mathsf{FinSet}^{\mathrm{iso}}$. Then objects are equivalent to integers $\mathbb{Z}_{\geq 0}$, and

$$\mathrm{Hom}(m, n) = \begin{cases} \emptyset, & m \neq n, \\ S_n, & m = n. \end{cases}$$

This category is *not* equivalent to the discrete category, since there is no way to faithfully map $S_n \to *$ for $n \geq 2$.

Example 2.3. On homotopy theory and natural transformations: consider the category $\mathsf{1\text{-}Trunc}$, where objects X are 1-truncated topological spaces, eg $\pi_k(X) = 0$ for $k > 1$. Morphisms are continuous maps up to homotopy. Then there is the category of groupoids where morphisms are functors. The claim is that these two functors are equivalent: a functor $\mathsf{1\text{-}Trunc} \to \mathsf{Grpd}$ is given by taking $\pi_{\leq 1}$, and we can construct topological spaces from groupoids for the inverse.

3 September 2, 2021

What is the point of abstraction? Sets allow us to not worry about the elements themselves (like apples or oranges), but about the structure of the set itself. Category theory is the next level of abstraction, where the objects have relations that keep track of structure.

There may be some constructions like exterior algebras, tensor products, etc that we wish to construct on finite dimensional vector spaces. Categorically, we can use the correspondence with the “skeleton” category $\mathsf{Vect}_{\mathrm{fd}}^{\mathbb{R}} \overset{\sim}{\longleftrightarrow} \mathsf{Euc}$, forming a **full subcategory**- a subcategory that loses some objects but remembers *all* morphism.

3.1 Initial and terminal objects

Recall the poset category $(P, \leq)$. We can ask for a *maximal* element $M \in P$ such that for all $x \in P$, $x \leq M$, or a *minimal* element m, where $m \leq x$ for all $x \in P$.

Definition 3.1. Let $\mathcal{C}$ be a category. We say $x \in \mathcal{C}$ is a **final object** (or *terminal object*) of $\mathcal{C}$ if for all $y \in C$, there exists a unique map $y \to x$. Similarly, an **initial object** $m \in \mathcal{C}$ is an object such that for all $y \in \mathcal{C}$, there exists a unique $m \to y$.

So in a poset, maximal elements are final and minimal objects are initial (if they exist). In general, universal properties are not guaranteed to exist: but if they do, they are unique.

Proposition 3.1. *If a final object exists, it is unique up to unique isomorphism. In other words, if M_1, M_2 are both final objects of $\mathcal{C}$, there is a unique $M_1 \xrightarrow{\sim} M_2$.*

Proof. Assume M_1 is final, so there exists a unique map $M_2 \to M_1$. Similarly, there exists a unique map $M_1 \to M_2$. Composing these two maps gives a unique map $M_1 \to M_1$, which must be the identity. ⊠

Example 3.1. Here are some examples of initial and final objects:

- In Set, the singleton $\{*\}$ is final while the empty set $\emptyset$ is initial.
- In $\mathsf{Vect}_{\mathbb{R}}$ or Ab, the zero vector space/trivial group is both initial (linear maps send identities to identities) and final.

  This gives rise to the idea of a **zero object**, which is both initial and final. “Geometric” categories like Top and Set tend not to have zero objects, but “linear” categories like the ones above do.
- In Ring, 0 is final while $\mathbb{Z}$ is initial (it has zero and one).
- The category of fields does not have an initial or terminal object.

3.2 Localization

We start with R a ring, and we choose a multiplicative subset $S \subseteq R$ (containing 1, closed under $\cdot$). We want to add inverse to $s \in S$ to the ring R. Consider the “undercategory” Ring todo:see other notes?? with objects rings T with a morphism $R \xrightarrow{\varphi_T} T$, with morphisms todo:??

Definition 3.2. The **localization** $S^{-1}R$ (if it exists) is the initial object of $\mathcal{C}_{R,S}$. In other words, todo:?

The way of constructing localizations by hand is by some equivalence relations $\frac{r}{s} \sim \frac{r'}{s'}$, which leads to $rs' - r's = 0$. However, there may be zero divisors, so instead mod out by $(rs' - r's)s'' = 0$, or $rs' - r's = \frac{0}{s''}$.

Note. Think of rings as function rings, with a space associated (eg smooth functions on $\mathbb{R}^n$).

3.3 Completion

Let R be a ring (eg $\mathbb{Z}$) and $I \subseteq R$ an ideal (p). todo:?? go back and finish Consider $R/I^2 \to R/I$. Completion is the limit on the left. Consider $\mathcal{C}$, the category of all rings T with compatible maps $T \to R/I^n$ for rall n.

Definition 3.3. The completion $\hat{R}_I$ is the final object of $\mathcal{C}$ if it exists.

4 September 7, 2021

Today we'll talk more about the theme of optimization problems in categories by universal properties. A more useful way of thinking about initial and final objects are limits and colimits.

4.1 Limits and colimits

Consider $\mathcal{C}$ a category, and we consider a functor $A\colon I \to \mathcal{C}$. We say A is an I**-shaped diagram**, where I is some small (simple) category. This functor takes the shape of I and makes a picture in $\mathcal{C}$.

$$\begin{tikzcd} i \arrow[rr] \arrow[dr] & & j \\ & k \arrow[ur] & \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} A_i \arrow[rr] \arrow[dr] & & A_j \\ & A_k \arrow[ur] & \end{tikzcd}$$

A **cocone** of $A\colon I \to \mathcal{C}$ is an object $B \in \mathcal{C}$ and maps $f_i : A_i \to B$ for all $i \in I$ commuting with all morphisms in I.

$$\begin{tikzcd} & A_j \arrow[dd] \arrow[dr] & \\ A_i \arrow[ur] \arrow[rr] \arrow[dr] & & B \\ & A_k \arrow[ur] & \end{tikzcd}$$

A **cone** is defined analogously, where the maps are from $B \to A_i$.

$$\begin{tikzcd} & & A_j \arrow[dd] \\ B \arrow[urr] \arrow[r] \arrow[drr] & A_i \arrow[ur] \arrow[dr] & \\ & & A_k \end{tikzcd}$$

Definition 4.1. A **colimit** of a diagram $A\colon I \to C$ is an initial object of cocones $\varinjlim A$. A **limit** of $A\colon I \to C$, $\varprojlim A$, is a final cone.

So colimits are "filling in" the diagram on the right, while limits are "filling in" the diagram on the left. This is one of the most important ideas in category theory, along with adjoint functors which essentially say the same thing.

Example 4.1. Let I be the category with two objects $\{1, 2\}$ and no maps between them. Then a functor $A\colon I \to \mathcal{C}$ gives two objects $A_1, A_2 \in \mathcal{C}$. To figure out the limit $\varprojlim A$, we need something mapping *to* the A_i. That is, a cone such that anything mapping into the A_i factors through $\varprojlim$. This turns out to be the **product** $A_1 \times A_2$.

$$\begin{tikzcd}
 & & A_1 \\
B \arrow[r, dashed] \arrow[urr] \arrow[drr] & \varprojlim := A_1 \times A_2 \arrow[ur] \arrow[dr] & \\
 & & A_2
\end{tikzcd}$$

Similarly, the **coproduct** is defined analogously.

$$\begin{tikzcd}
A_1 \arrow[dr] \arrow[drr] & & \\
 & \varinjlim := A_1 \coprod A_2 \arrow[r, dashed, "\exists!"] & B \\
A_2 \arrow[ur] \arrow[urr] & &
\end{tikzcd}$$

In Set, let X, Y be sets. Then for a set C with maps $C \xrightarrow{f} X$ and $C \xrightarrow{g} Y$, the maps factor through the unique map (f, g) by projection. In the same way, for maps $X \xrightarrow{h} B, Y \xrightarrow{k} B$, they factor through the disjoint union by inclusion.

$$\begin{tikzcd}
 & & X \arrow[dr, "\iota_X"] \arrow[drr, "h"] & & \\
C \arrow[urr, "f"] \arrow[drr, "g"'] & \varprojlim := X \times Y \arrow[l, dotted, "\exists!"'] \arrow[ur, "\pi_X"] \arrow[dr, "\pi_Y"'] & & \varinjlim := X \coprod Y \arrow[r, dashed, "\exists!"] & B \\
 & & Y \arrow[ur, "\iota_Y"'] \arrow[urr, "k"'] & &
\end{tikzcd}$$

Example 4.2. In Grp, the product is just $G \times H$, but the coproduct is actually the free product. This illustrates the idea that in general colimits are more complicated for limits.

Consider the forgetful functor For: Grp $\to$ Set, where $G \mapsto$ the underlying set and $\varphi: G \to H$ maps to the underlying set map. The functor For takes products to products, but *doesn't* preserve coproducts. The general principle is that forgetful functors preserve limits. A mneumonic is that limits $\longleftrightarrow$ subs and colimits $\longleftrightarrow$ quotients, and quotients are more complicated. Limits map *in*, and colimits map *from*.

Example 4.3. Consider the categories Vect_k or R-Mod. It turns out that products and coproducts are the same, the direct product $V \oplus W$. The category Ab is different from the category of groups, because the direct some $M \oplus N$ is a biproduct while for groups the product $M \times N$ and the free product $M * N$ are very different. We defined the (co)product by the (co)limit of two points, and similarly the repeated direct sum $\bigoplus_i V_i$ is the (co)limit of a bigger diagram.

In Vect, we have infinite products and infinite coproducts. But they are different. The product of vector spaces $\prod V_i = V_1 \times V_2 \times \cdots x \cdots$ is the set theoretic $\{(v_1, v_2, \cdots)\}$, while the infinite direct sum is $\bigoplus V_i = \{(v_1, v_2, \cdots) \mid$ all but finitely many $v_i = 0\}$..

4.2 Fiber products/pullbacks

Consider our index category $\bullet_1 \to \bullet_2$ with a functor to $\mathcal{C}$. This gets send to $A_1 \to A_2$. A colimit would be the best thing that A_1,A_2 map to in a compatible way. $A_1 \to A_2$, and $A_2 \to A_2$ by id, so $\varinjlim\{A_i\} = A_2$, similarly $\varprojlim\{A_i\} = A_1$. This category has a final and initial project, then the colimit $\varinjlim A = A$(final object), similarly $\varprojlim A = A$(initial object).

Now consider the diagram $\bullet_1 \to \bullet_3 \leftarrow \bullet_2$. This diagram has a final object but no initial object. We ask for the limit (???) todo:figure out limit or colimit, initial or terminal, diagram

The **pushout** is defined in a similar way, as the *colimit* of the diagram $\bullet_1 \leftarrow \bullet_3 \to \bullet_2$. todo:diagram

set:disjoint union mod equiv relation, in grp its frere product with amalgamation.

Our next class of examples is (co)equalizers.

localization and complection as limits and coliimits?

5 September 9, 2021

Last time we talked about limits and colimits, some examples being products and coproducts, fiber products, equalizers and coequalizers, completion and localization. To describe finite limits and colimits in Set, in some sense you only need to understand products and coproducts, equalizers and coequalizers.

Consider a diagram of sets, where $\{A_i\}_{i\in I}$ sets, $\{A_i \xrightarrow{f_{ij}^k}\}$. The basic limit is the product of the A_i's, $\prod A_i$. This is the limit of two (or n) dots without arrows between them. To justify the slogan "limits are subs, colimits are quotients", $\varprojlim\{A_i\}$ is a subset of $\prod A_i$. We have

$$\varprojlim\{A_i\} = \{(\cdots, a_i, a_j, \cdots) \in \prod A_i \mid f_{ij}^k(a_i) = a_j \text{ for all } i, j, k\}.$$

An exercise is to check that this satisfies the universal property. For the colimit, the claim is that $\varinjlim A_i = \coprod A_i/\sim$, the question is now "what is this equivalence relation?" The relation is described by $\sim$: the minimal equivalence relation containing $a_j \sim f_{ij}^k a_i$, $A_i \xrightarrow{f_{ij}^k} A_j$.

We can think of **filtered diagrams** as "trying to have" a final object. Filtered diagrams have "direction": this means that for any two arrows t_1, t_2, they eventually compose to a common place. Filtered diagrams typically don't have a final object as this would make the category uninteresting.

Definition 5.1. A diagram I is **filtered** if for all $A, B \in I$, there exists a C such that

$$\begin{tikzcd} A \arrow[dr] & \\ & C \\ B \arrow[ur] & \end{tikzcd}$$

and any two morphisms

$$\begin{tikzcd} A \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] \arrow[rr, bend left] \arrow[rr, bend right] & B \arrow[r, dashed] & C \end{tikzcd}$$

have coequalizers.

The colimit is "trying to be the root of the tree", or $\varinjlim_I \{A_i\}_{i\in I} = \coprod A_i / a_i \in A_i \sim A_j \in A_j$ if there exists morphisms identifying a_1, a_2. When we say "morphisms identifying", this precisely means there exists

$$\begin{tikzcd} a_i \in A_i \arrow[dr, "g_1"] & \\ & A_N \\ a_j \in A_j \arrow[ur, "g_2"'] & \end{tikzcd}$$

such that $g_1(a_i) = g_2(a_j)$. In general, filtered colimits behave much nicer with functors than other colimits.

5.1 Adjoints

Recall the method of least squares from linear algebra, where we need to define an inner product to define projections, giving a best approximation of v that lives in w (for $v \in V, w \in W, W \subseteq V$). This is characterized by saying $\langle \mathrm{proj}_W w, w\rangle$ for $w \in W$ is equal to $\langle v, w\rangle$. Denote the inclusion by $i: W \hookrightarrow V$, so $\langle \mathrm{proj}_W, v, w\rangle_W = \langle v, i(w)\rangle_V$.

We generalize this notion in linear algebra by an adjoint map. Consider a map of inner product spaces $f : W \to V$. Then $f^\dagger : V \to W$ is "the best inverse" for f, where $f^\dagger$ is characterized by

$$\langle f^\dagger v, w\rangle_W = \langle v, f w\rangle_V.$$

Here is a schematic way of looking at it.

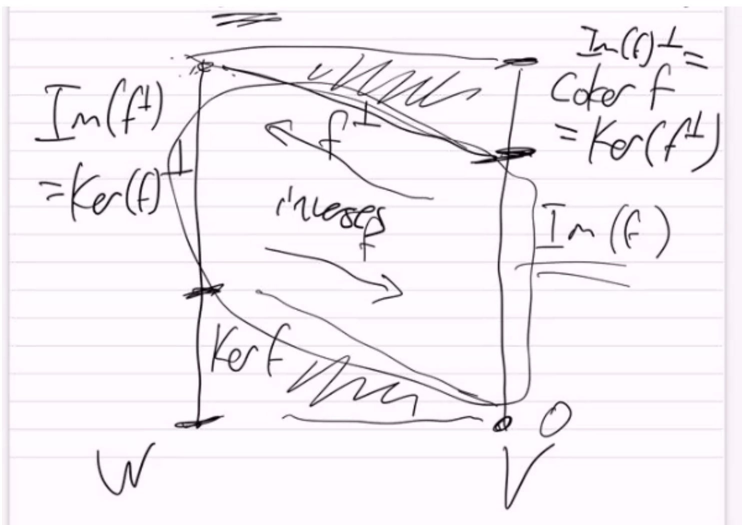

Figure 1: Visualizing adjoints in linear algebra.

The kernel is collapsed to zero, and the cokernel is the remainder of the image. Then $\mathrm{im}\, f^\dagger$ is the orthogonal complement of $\ker f$ (which makes sense with an inner product), and f and $f^\dagger$ are inverses. This lifts the statement $\dim W - \dim \ker = \dim \mathrm{im}\, f$ to a statement about the spaces themselves.

Now we try to do the same thing in category theory. Instead of spaces, we have a functor between categories $F : \mathcal{C} \to \mathcal{D}$. The most crucial piece of data we are missing is the inner product. Given two objects $A, B \in \mathcal{C}$, we think of the pairing $\mathrm{Hom}_{\mathcal{C}}(A, B)$ as an inner product of sorts, a way of measuring the "distance" between the two. This is a non-symmetric inner product.

Definition 5.2. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. We say a functor $G : \mathcal{D} \to \mathcal{C}$ is a **right adjoint** to F if $\mathrm{Hom}_{\mathcal{C}}(A, G(B)) \simeq \mathrm{Hom}_{\mathcal{D}}(F(A), B)$, where $A \in \mathcal{C}$, $B \in \mathcal{D}$ "functorially in A, B". We say F is a **left adjoint** to G.[3]

$\mathrm{Hom}_{\mathcal{C}}(-, G(-)) \simeq \mathrm{Hom}_{\mathcal{D}}(F(-), -)$ should be a natural isomorphism of functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathsf{Set}$. To make sense of that last statement, let $\mathcal{C}$ be a category. Then $\mathrm{Hom}_{\mathcal{C}}(-, -) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{Set}$ is a functor.

Notation. If (F, G) are an adjoint pair, then G is the right adjoint and F is the left. We also use the notation $F \dashv G$ which comes from turning the perpendicular ($\perp$) sign sideways, since the relation is not symmetric.

Example 5.1. todo:free and forgetful functor

[3] To remember this, note that G is appearing on the right side of the Hom's and F is appearing on the left side of the Hom's.

Example 5.2 (Tensor products)**.** Suppose we have two vector spaces V, W and we want to define $V \otimes W$. Then for a bilinear map $\varphi : V \times W \to U$, the tensor product satisfies by a universal property

$$\begin{tikzcd} & V \otimes W \arrow[dr, dotted, "\exists!"] & \\ V \times W \arrow[ur] \arrow[rr, "\varphi"] & & U \end{tikzcd}$$

or $\varphi(v, w) \in U$, todo:STOP SCROLLING BY SO FAST

6 September 14, 2021

todo:? missed first 10 min The idea that adjoints and limits are the same thing was advertised, let us make this precise. todo:?

6.1 The Yoneda lemma

The Yoneda lemma says that the Hom pairing on any category $\mathcal{C}$ is nondegenerate. If we have V a finite dimensional vector space with a nondegenerate inner product, then

$$V \xrightarrow{\sim} V^* = \mathrm{Hom}(V, \mathbb{R})$$

is an isomorphism. In the infinite-dimensional setting, consider $V = C_c^\infty(\mathbb{R})$, the smooth compactly supported functions on $\mathbb{R}$. Then the dual V^* is a distribution denoted $C^{-\infty}(\mathbb{R})$. todo:some other stuff about functional analysis

Let $\mathcal{C}$ be a category, and $\hat{\mathcal{C}} := \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$. Let $X \in \mathcal{C} \mapsto$ the functor represented by X, $h_X \in \mathcal{C}$, $h_X = \mathrm{Hom}_{\mathcal{C}}(-, X)$. Is h_X a functor?

todo:?

Corollary 6.1. *Localization is a tensor product.*

Let M, N be R-modules, $S \subseteq R$. Consider $S^{-1}(M \otimes_R N) \simeq S^{-1}M \otimes_R S^{-1}N$. Left adjoints and colimits commute. As an example, $S^{-1}M = S^{-1}(R \otimes_R M)$, $[M = R \otimes_R M]$.

6.2 End of category theory

7 September 16, 2021

Welcome to algebraic geometry! Today we start talking about algebraic geometry.

7.1 Motivating schemes

What's the idea behind schemes? We want to take the theory of rings and embed them in a contravariant fashion, identifying them with something called affine schemes by Spec ($\mathsf{Rings}^{\mathrm{op}} \underset{\mathrm{Spec}}{\xrightarrow{\sim}} \mathsf{Aff}$). It turns out affine schemes are equivalent to rings as categories, so sometimes people define affine schemes as $\mathsf{Rings}^{\mathrm{op}}$, which is kind of cheating, since we want affine schemes to be some kind of geometric object. We think of **affine schemes** as a subcategory of a bigger class of geometric objects, which we call **schemes** Sch. One can make the comparison that affine schemes are to schemes as Euclidian spaces are to manifolds. We build schemes by gluing together affine schemes, but unlike the theory of smooth manifolds where Euclidian spaces are all isomorphic to $\mathbb{R}^n$, the theory of affine schemes is extremely rich (equivalent to the theory of rings).

What do we mean by a "geometric object"? Let LRS denote the notion of a **locally ringed space**, which is a set plus a topology, and a notion of a function. (We postpone this idea for a little while.) Schemes are going to be sets, with a topology and a function, and so also locally ringed spaces. Given a LRS, we can attach its ring of (global) funtions which defines a functor LRS $\xrightarrow{\text{global functions}} \mathsf{Rings}^{\mathrm{op}}$. Then the functor Spec is going to be the right adjoint to the functor of global functions.

$$
\begin{tikzcd}
\mathsf{Rings}^{\mathrm{op}} \arrow[r, "\sim", "\mathrm{Spec}"'] \arrow[rrr, bend right, "\mathrm{Spec}"'] & \mathsf{Aff} \arrow[r, phantom, "\subset"] & \mathsf{Sch} \arrow[r, phantom, "\subset"] & \mathsf{LRS} \arrow[lll, bend right, "\text{global functions}"']
\end{tikzcd}
$$

7.2 The spectrum of a ring

Our goal for today is to define Spec(R) as a set. What should points of Spec(R) be? Let's start with some classical algebraic geometry. In an ideal world, a prerequisite for this class would be a course in classical algebraic geometry, the theory of varieties (Harris). Harris' book is extremely classical and heavy with examples (the foil to this class). Classically, what people mean by "affine variety" is a subset of n-dimensional affine space over an algebraically closed field of characteristic 0 (say $\mathbb{C}$). Look at a subset of $\mathbb{C}^n \supset V$ which is cut out by polynomial equations, look at their common zero locus, and that is a variety. Consider the quotient of a polynomial ring $\mathbb{C}[x_1, \cdots, x_n]/I$ where I is the ideal of functions that vanish on V - this is the setting of classical algebraic geometry.

In classical algebraic geometry, there is a natural idea of what points should do. There is no ambiguity about what a point is, since they're subsets of $\mathbb{C}^n$ that solve a given set of equations. The big idea is Hilbert's **Nullstellensatz** (the zero space theorem in German), which says that points are given by maximal ideals. If $\mathfrak{m} \subseteq R$ is maximal, then R/M is a field. So $\mathfrak{m} \subseteq R$ is equivalent to a surjective homomorphism $R \twoheadrightarrow k$. Why is this relevant to a point? Points give maps to fields, where we evaluate functions at points (the evaluation homomorphism) giving a map $R \twoheadrightarrow \mathbb{C}$ in this case. We need to know the following: are there other weird maximal ideals we forgot to consider?

Hilbert's Nullstellensatz. *If k is a field, maximal ideals $\mathfrak{m}$ in $R = k[x_1, \cdots, x_n]$ have residue field $R/\mathfrak{m}$ a finite extension of k.*

8 September 21, 2021

9 October 5, 2021

Our first in person lecture! What are sheaves? They are

- substitutes for function spaces,
- a geometric inerpretation for modules.

For $R \mapsto \mathrm{Spec} R \ni \{\lambda \colon R \to k\}$ equal to eigenvalues for (commutry?) operation on V/k. $r \in R \mapsto r \cdot v = \lambda(r) v$, so eigenspaces are sheaves on Spec$R$.

To define sheaves, we need X a topological space, $\mathcal{C}$ a category (usually $\mathsf{Set}, \mathsf{Ab}$, or Ring). A $\mathcal{C}$-valued sheaf on X is a *local* expression on X generalizing propereties of functions. For $U \subseteq X$ open, this leads to $F(U) \in \mathcal{C}$ "functions on U".

9.1 Presheaves

Definition 9.1. For X a topological space, let $\mathrm{Top}(X) = \mathrm{Open}(X)$ be the poset of open sets in X under inclusion.

Definition 9.2. A ($\mathcal{C}$-valued) **presheaf** is a functor $F\colon \mathrm{Open}(X)^{\mathrm{op}} \to \mathcal{C}$, i.e., $U \subseteq X \mapsto F(U) \in \mathcal{C}$, and $V \subseteq U \mapsto \mathrm{res}_U^V\colon F(U) \to F(V)$, where res is the restriction.

For F, G, $\mathcal{C}$-valued presheaves, $\varphi\colon F \to G$ means for $V \subseteq U$, the following diagram commutes:

$$\begin{tikzcd} \varphi_U\colon F(U) \arrow[r] \arrow[d, "\mathrm{res}"] & G(U) \arrow[d, "\mathrm{res}"] \\ \varphi_V\colon F(V) \arrow[r] & G(V) \end{tikzcd}$$

In other words, a natural transformation. Denote the category of presheaves by $\mathcal{C}_X^{\mathrm{pre}}$.

Example 9.1. We have the *constant presheaf*, where for $A \in \mathcal{C}$, $A(U) = A$, and res = id. We also have the *skyscraper* presheaf, where for $x \in X$, $\mathcal{C} = \mathsf{Ab}$,

$$\delta_{A,x}(U) = \begin{cases} A & \text{for } x \in U, \\ 0 & \text{for } x \notin U. \end{cases}$$

Example 9.2. The important examples of *functions* of any kind. For $X = \mathbb{R}^n$, $\mathcal{C} = \mathsf{Set}, \mathsf{Ab}, \mathsf{Ring}$, then $C(U)$ is the space of continuous $\mathbb{R}$-valued functions, C^∞ means smooth, C^k means k-differentiable. The complex analytic functions on CP^1 are all constant– in general functions don't extend. So instead of functions on the whole space, we keep track of functions on smaller spaces.

Example 9.3. Fix a topological space Y (for example $\mathbb{R}$), and consider $\mathrm{Map}_{\mathrm{cont}}(X, Y)$, the continuous maps $X \to Y$. This forms a presheaf by $U \mapsto \mathrm{Map}_{\mathrm{cont}(U,Y)}$.

Let's formalize the global and local aspects of a presheaf.

Global	**Local**
Global sections of F, $\Gamma(F) := F(X)$.	Stalks for $x \in X$
Here $F(U) :=$ sections on U.	Germs

Figure 2: Global and local aspects of presheaves.

Definition 9.3. Define the **stalk of** x by $F_x := \varinjlim_{x \in U} F(U)$. We can think of this by the value of F on the infinite intersection of opens $F\left(\bigcap U\right)$, but infinite intersection of opens don't make sense, hence our notion.

For $\mathcal{C} = \mathsf{Ab}, \mathsf{Set}$, we identify this with the set $\{f_U \in F(U), x \in U\}/f_U \sim f_V$ if they have the same restriction to $W \subseteq U \cap V$. This equivalence relation is the same as modding by germs of functions. Stalks and germs are talking about the same thing, we usually talk about the stalk of a sheaf and the germ of a function. todo:something about presheavse as taylor series with nonzero radius of convergence on C^∞?

9.2 Sheaves

Now we want to introduce a notion of gluing/descent/patching to make sheaves. Let $U = \bigcup_{i \in I} U_i$ be a covering of U, then functions on U correspond to functions on U_i which match on overlaps. For f on U, we have f_n on U, where $\mathrm{res}_U^{U_i} = f|_{U_i}$. If $U_{ij} = U_i \cap U_j$, then $f|_{U_i}\big|_{U_i \cap U_j} = f|_{U_j}\big|_{U_i \cap U_j}$.

Definition 9.4 (First definition of a sheaf)**.** For $F \in \mathcal{C}_X^{\text{pre}}$, we say F is a **sheaf** if

- *identity*: For a cover $U = \bigcup_{i\in I} U_i$, $F(U) \xrightarrow{\prod \text{res}_U^{U_i}} \prod F(U_i)$ is a monomorphism (or injective).
- *gluing*: Given sections $\{f_i \in F(U_i)\}$ agreeing on overlaps, $f_i|_{U_{ij}} = f_j\big|_{U_{ij}}$ implies there exists an $f \in F(U)$ with $f = f|_{U_i}$.

We give a little bit more formal definition for those who are into that sort of thing.

Definition 9.5. We say F is a **sheaf** if for $U = \bigcup_{i\in I} U_i$,

$$F(U) \to \prod_{i\in I} F(U_i) \rightrightarrows \prod_{i,j\in I} F(U_{ij})$$

$\{f_i\}_{i\in I}$, $f_i|_{U_{ij}} = f_j\big|_{U_{ij}}$ is an *equalizer*.

This is a lot more compact than the previous definition.

Example 9.4. We have $F(U \amalg V) \cong F(U) \times F(V)$ for sheaves.

- Functions of any kind form sheaves.
- Maps to a fixed target form a sheaf.
- Let $\mathcal{C} = \mathsf{Set}$, $X =$ a topological space (say S^1), and $F \in$ sheaves of sets on X. Let $F(U)$ be the sectionsof $\pi\colon Z \to X$ over U. For $Z \xrightarrow{\pi} X$ (eg the universal cover $\mathbb{R} \to S^1$), we look at the **sheaf of sections** of $\pi\colon F(U) =$ sections of $\pi^{-1}(U) \to U$, where a section is a map $s\colon U \to \pi^{-1}(U)$ such that $\pi s = \text{id}$. These form a presheaf, and we need to check that these form a sheaf. This is very much like a map since we can glue in the same way, in fact sections of π are equivalent to maps $X \to Y$. This is the notion of a *graph* of a function, a section of the projection $X \times Y \to X$.

  Here $F(S^1) = \emptyset$, but $F(U) \simeq \mathbb{Z}$ for U an interval. There are no global sections ($\Gamma(F) = \emptyset$), but there are tons of local sections. This is why sheaves are useful.
- Recall the constant presheaf, $A = \mathbb{Z} \in \mathsf{Ab}$, $\mathbb{Z}(U) = \mathbb{Z}$. Something is off– the constant presheaf is not actually a sheaf. Given a disjoint union, if the constant presheaf were a sheaf by the sheaf axiom we would have $U \amalg V \mapsto A \times A$ for $U \mapsto A, V \mapsto A$, however for U, V disconnected this would fail.

  Define the **constant sheaf** $A^{\text{sheaf}}\colon U \mapsto A$ if U is connected. Then $U \mapsto \prod A$ where each A is a component of U. Think of A as a set, and $\text{Map}_{\text{cont}}(U, A)$. We'll see next time there is a way to correct every presheaf.

Our example $\mathbb{R} \to S^1$ earlier is an example of a **locally constant** sheaf, which looks like $\mathbb{Z}$ on small enough open sets. The fact that the circle has an interesting topology is captured by the fact that locally it's doing nothing but has some sort of global structure.

10 October 7, 2021

What lecture are we actually on?

10.1 More on sheaves

Recall that for $X = \operatorname{Spec} R$ a topological space, a (structure) sheaf $\mathcal{O}_X$ is a "notion of functions". We have $\mathcal{O}_X : U \subseteq X$ open $\mapsto \mathcal{O}_X(U) \in \mathsf{Ring}$, then a presheaf $V \subseteq U$ implies that $\text{res}_U^V\colon \mathcal{O}_X(U) \to \mathcal{O}_X(V)$ is associative. For a sheaf, $f \in \mathcal{O}(U)$ implies that f is determined by $f|_{U_i}$, where the U_i cover U. So $\mathcal{O}_X(U) \hookrightarrow \prod \mathcal{O}_X(U_i)$. Given $f_i \in \mathcal{O}(V_i)$ agreeing on $U_{ij} = U_i \cap U_j$, then this comes from $f \in \mathcal{O}(U)$.

Recall that $\mathcal{C}_X^{\text{pre}}$ is the category of $\mathcal{C}$-valued pressheaves on X, which contains the category of sheaves $\mathcal{C}_X$. Fix $A \in \mathsf{Set}$, and let $A^{\text{pre}} \in \mathsf{Set}_X^{\text{pre}}$ be the constant presheaf with value A, or $A^{\text{pre}}(U) = A$, res $=$ id. Let A be the constant sheaf in Set_X, which has some value (A) on "small enough" opens. Then

$$A(U) = A\text{-valued closed functions on } U = \mathrm{Map}_{\text{cts}}(U, A^{\text{disc}}).$$

So $A(U) = A$ if U is connected, and if $U = U_1 \amalg U_2$, $A(U) = A \times A$. These functions have a name, being *locally constant A-valued functions on U*.

Sheaves are given by local conditions. All $\mathbb{R}$-valued functions on X form a sheaf, contained in the continuous, then differentiable (and analytic), then smooth functions. Recall sections of a map $\pi \colon Z \to X$, $s \colon X \to Z$ with $\pi s(x) = x$. Then for $U \subseteq X$, *sections over U* map $U \xrightarrow{s} Z$ with $\pi s(x) = x$ for $x \in U$. For every topological space Y, the sheaf of sets $U \mapsto \mathrm{Map}_{cts}(U, Y)$ is a Y-valued function. This implies that sections F of $Z \to X$ are also sheaves: $F(U) = \{s \colon U \to Z,\ \pi s = \mathrm{id}\} \subseteq \mathrm{Map}(U, Z)$. We need to check the identity axiom:

$$\begin{tikzcd}
\mathrm{Sect}(U,Z) \arrow[rr, hook] \arrow[dr, dotted, hook] & & \mathrm{Maps}(U,Z) \arrow[r, hook] & \prod \mathrm{Map}(U_i, Z) \\
& \prod \mathrm{Sect}(U_i, Z) \arrow[ur, hook] & &
\end{tikzcd}$$

So

$$\begin{tikzcd}
\mathrm{Sect}(U,Z) \arrow[r, hook] \arrow[d] & \mathrm{Maps}(U,Z) \arrow[d] \\
\prod \mathrm{Sect}(U_i, Z) \arrow[r, hook] & \prod \mathrm{Maps}(U_i, Z)
\end{tikzcd}$$

So sections form a sheaf. Why are continuous maps $\mathrm{Map}_{\text{cts}}(U, Y)$ a sheaf? Why do continuous maps glue? Consider

$$\mathrm{Hom}_{\mathsf{Top}}\left(\bigcup_{i \in I} U_i, Y\right) = \mathrm{Hom}_{\mathsf{Top}}\left(\coprod U_{ij} \rightrightarrows \coprod U_i \to U, Y\right) = \varinjlim \mathrm{Hom}_{\mathsf{Top}}(U_i, Y).$$

This implies that

$$\mathrm{Hom}(U, Y) \to \prod \mathrm{Hom}(U_i, Y) \rightrightarrows \mathrm{Hom}(U_{ij}, Y).$$

10.2 Sheaves vs presheaves

For F a *sheaf*, F is determined by $F(U_\alpha)$ a basis of opens. Then $U = \bigcup_{\alpha \in \text{ basis}} U_\alpha$. If $X = \operatorname{Spec} R$, $\mathcal{O}_X$ is the sheaf of rings on X. Then for $f \in R$, $\mathcal{O}_X\left(D_f = \operatorname{Spec} f^{-1}R\right) = f^{-1}R$. Recall for $x \in X$, we have the notion of the stalk of a presheaf F, where $F_x = \varinjlim_{x \in U} F(U) = \{x \in U_\alpha, s_\alpha \in F(U_\alpha)\}$, which is "equal" to $F\left(\bigcap_{x \in U} U\right)$. For $x \in U$, we have a map $F(U) \to F_x$. Putting these together, we have a map $F(U) \to \prod_{x \in U} F_x$. The claim is that the identity axiom for a sheaf forces this map to be injective. To see this, $s \in F(U) \mapsto \{s_x\}_{x \in U} \in \prod_{x \in U} F_x$. Now consider $F(U) \cong \left\{\prod_{x \in U} s_x \textbf{ compatible} \text{ families } \in \prod F_x\right\}$, where $\prod s_x$ compatible if for every x there is a representative $\{x \in V \subseteq U, s_V\}$ of s_x such that $s_V | Y = s_Y$ for all $y \in V$.

Theorem 10.1. *There exists a* ***sheafification*** *function which is left adjoint to the inclusion*

$$\begin{tikzcd}
\mathcal{C}_X \arrow[r, hook] & \mathcal{C}_X^{\text{pre}} \arrow[l, bend right, "(-)^{\text{sh}}"']
\end{tikzcd}$$

where $F \mapsto F^{\text{sh}}$, i.e., $\mathrm{Hom}_{\mathcal{C}_X}(F^{\text{sh}}, \mathcal{G})$ *(where $\mathcal{G}$ is a sheaf in both categories)* $= \mathrm{Hom}_{\mathcal{C}_X^{\text{pre}}}(F, \mathcal{G})$. *This is equivalent to the* *following diagram:*

$$\begin{tikzcd}
\overset{\text{presheaf}}{F} \arrow[rr] \arrow[dr] & & \overset{\text{sheaf}}{\mathcal{G}} \\
& F^{\text{sh}} \arrow[ur, dotted, "\exists!"'] &
\end{tikzcd}$$

This implies that if F is a presheaf, then $F^{\text{sh}} = F$, or $(A^{\text{pre}})^{\text{sh}} = A$. So the construction of stalks of F and F^{sh} are the same.

11 October 12, 2021

Our plan for today is to try to finish off sheafs, then next time we'll get back to schemes. For $\mathcal{C} = \mathsf{Set}, \mathsf{Ab}, \mathsf{Ring}, \ldots$, and X a topological space, we consider $\mathcal{C}_X$ the $\mathcal{C}$-valued sheaves on X. Then $x \in X \rightsquigarrow F_x \in \mathcal{C}$. So presheaves give $x \mapsto F_x$ a $\mathcal{C}$-valued function on X. A sheaf is a continuous/locally constant $\mathcal{C}$-valued function, so a sheaf is determined by its stalks. We showed last time that for a sheaf F, $F(U) \hookrightarrow \prod_{x \in U} F_x$.

For F a presheaf, we attached the following thing called $F^{\text{sh}}(U)$, called the **sheafification** of F. Define

$$F^{\text{sh}}(U) := \left\{ (s_x) \in \prod F_x \mid \forall x \in U,\ \exists \text{open nbd } x \in V \subseteq U \text{ and } s_v \in F(V) \ni \forall y \in V, s_y = \text{germ of } s_V \text{ at } y \right\}.$$

This is some kind of continuity condition.

Theorem 11.1. *The map $F \mapsto F^{\text{sh}}$ defines a functor $\mathcal{C}_X^{\text{pre}} \to \mathcal{C}_X$ from presheaves to sheaves, which is left adjoint to the inclusion $\mathcal{C}_X \hookrightarrow \mathcal{C}_X^{\text{pre}}$.*

Proof. If $\mathcal{G}$ is a sheaf rather than a presheaf, then $G \xrightarrow{\sim} \mathcal{G}^{\text{sh}}$ canonically. There is a natural map $F(U) \to F^{\text{sh}}(U)$ by attaching honest sections, which is a map $F \to F^{\text{sh}}$. Then the adjunction statement makes the following diagram commute:

$$\begin{tikzcd}
F \arrow[rr, "\varphi"] \arrow[dr] & & G \\
& F^{\text{sh}} \arrow[ur, dotted, "\exists!"'] &
\end{tikzcd}$$

To see this, consider the following diagram:

$$\begin{tikzcd}
F(U) \arrow[rr, "\varphi"] \arrow[dr] \arrow[dd] & & G(U) \arrow[dr, "s"] \arrow[dd] & \\
& F^{\text{sh}}(U) \arrow[ur] \arrow[rr] \arrow[dl] & & G^{\text{sh}}(U) \arrow[dl] \\
\prod_{x \in U} F_x \arrow[rr] & & \prod_{x \in U} G_x &
\end{tikzcd}$$

⊠

Example 11.1. We have (constant presheaf)$^{\text{sh}}$ =constant sheaf. For A a set, $A^{\text{pre}}(U) = A$ for all U. The constant sheaf $A(U)$ is equal to locally constant A-valued functions.

The map $F \mapsto \{F_x\}_{x \in X}$ is a functor from sheaves of sets to products of sets over $x \in X$. The claim is that this functor is *faithful*, or injective over Hom's. So a map of sheaves $\varphi : F \to G$ is determined by what it does on stalks

$\{\varphi_x : F_x \to G_x\}$. This is completely false for presheaves, going to stalks loses a lot of information.

$$\begin{tikzcd} F(U) \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "\psi"'] \arrow[d, hook] & G(U) \arrow[d, hook] \\ \prod F_x \arrow[r, shift left, "\varphi_x"] \arrow[r, shift right, "\psi"'] & \prod G_x \end{tikzcd}$$

Passing to stalks reflects monomorphisms and epimorphisms, i.e., $\varphi : F \to G$ is monic/epic if $\varphi_x : F_x \to G_x$ is for all x. In fact, $F \xrightarrow{\varphi} G$ is monic/epic/an isomorphism iff the φ_x for all $x \in X$. So there is a tight categorical connection between properties of sheaves and on their stalks. Things get interesting in the middle– what do we mean by that? We have maps on sheaves $F \to G$, then maps on stalks $F_x \to G_x$, and in the middle maps on opens $F(U) \to G(U)$. todo:?

Example 11.2. Consider $\mathcal{C} = \mathsf{Ab}$, and write down some short exact sequences $0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$ where φ is injective, ψ is surjective, and $C = B/A$. We will try to define this in the world of sheaves:

$$\begin{tikzcd} \ker\varphi \arrow[r] \arrow[d] & B \arrow[d, "\varphi"] \\ 0 \arrow[r] & C \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r] \arrow[d, "\varphi"] & 0 \arrow[d] \\ B \arrow[r] & \operatorname{coker}\varphi \end{tikzcd}$$

For $O \in \mathsf{Ab}_X$, O a sheaf, $O(U) = O$ for all U. Consider sheaves $\mathsf{Ab}_{\mathbb{C}}$ on $\mathbb{C}$ or $\mathsf{Ab}_{\mathbb{C}^*}$ on $\mathbb{C}^*$. Here $\mathbb{Z} \to \mathcal{O}$, where $\mathcal{O}$ is the sheaf of holomorphic/analytic (or smooth) functions. Constant functions are nice, so we have an inclusion $0 \to \mathbb{Z} \to \mathcal{O}$. What is the cokernel? Given f a function, attach to it $e^{2\pi i f}$ which is a nowhere vanishing function, where for f integer valued $e^{2\pi i f} = 1$. Then

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \xrightarrow{e^{2\pi i(-)}} ? \longrightarrow 0$$

Here ? is the abelian group of nonvanishing functions. Define $G(U) - \{e^{2\pi i f}, f \in \mathcal{O}(U) = \operatorname{im}\left(e^{2\pi i -}\right)\}$. The claim is that this is a presheaf but not a sheaf– why? For $\mathsf{Ab}_{\mathbb{C}^*}$, consider the function $g = z$ nowhere vanishing but is *not* of the form $e^{2\pi i f}$ for $f \in \mathcal{O}(\mathbb{C}^*)$, which just means we can't find a branch of the logarithm for it. Every nowhere vanishing function is locally in the image of $e^{2\pi i -}$. Analogously,

$$0 \longrightarrow \mathbb{Z}_X \longrightarrow C_X^\infty \xrightarrow{e^{2\pi i x}} \{\text{smooth } S^1\text{-valued functions}\}_X \longrightarrow 0$$

where $X = S^1$, which reflects the interesting topology of the circle. Now consider

$$\pm 1 \longrightarrow \{\text{nowhere vanishing functions}\} \xrightarrow{(-)^2} \{\text{nowhere vanishing square functions}\},$$

which wants to look like this:

$$\mathbb{Z}/n \hookrightarrow \mathcal{O}^* \twoheadrightarrow \mathcal{O}^*.$$

On $\mathbb{C}^*$, $\sqrt{z}$ does not exist, since the sheafification $z \in \mathcal{O}^*$ is not in the image of $(-)^2$. Topologically, this says we have a double cover of the circle. Now for $X = S^1$, consider

$$0 \to \mathbb{R} \to C^\infty \xrightarrow{d} \Omega^1_{\mathrm{cl}} \to 0$$

a short exact sequence of sheaves, where Ω^1_{cl} is the set of closed 1-forms. The derivative $df = 0$ implies f is locally constant. This is precisely the Poincaré lemma, which says that locally for any ω with $d\omega = 0$, $\omega = df$ for some f. In other words, $\Omega^1_{\mathrm{cl}} = \Omega^1_{\mathrm{ex}}$ on small enough opens, so there is a presheaf $\Omega^1_{\mathrm{ex}} = \operatorname{im}(d)$ since exactness is not a local condition. The discrepancy between the sheaf Ω_{cl} and the presheaf Ω_{ex} is de Rham cohomology.

12 October 14, 2021

Our plan for the next two lectures is to finally discuss schemes. For a ring R, we defined $\operatorname{Spec} R$ as the set of prime ideals inside R. Then define a topology by the closed subsets V_S for $S \subseteq R$, where V_S is the vanishing locus of all primes containing S. A special example is V_f for $f \in R$, where $V_S = \bigcap_{f \in S} V_f$. This is equivalent to $D_f = \operatorname{Spec} R \setminus V_f$, where D_f is the nonvanishing locus of f. Then open $U = \bigcup_{f \text{ nonvanishing}} D_f$. Another key example is $\mathcal{O}_{\operatorname{Spec} R}$, which is a sheaf of rings on $\operatorname{Spec} R$. Our goal today is to construct this structure sheaf.

Inside of $\operatorname{Spec} R$, we want $\mathcal{O}(\operatorname{Spec} R) = R$. To find $\mathcal{O}(D_f)$, observe that as a topological space $D_f \cong \operatorname{Spec} f^{-1}R = R_f$ where R_f is the localization of R by f. Functions in scheme theory are not uniquely determined by their vanishing locus. Recall $f^{-1}R \simeq S_f^{-1}R$, where S_f is equal to all elements $g \in R$ such that the vanishing locus $V(g) \subseteq V(f)$, or $D_g \supset D_f$. This is equal to all functions vanishing nowhere on $U = D_F$. Define $\mathcal{O}'_{\operatorname{Spec} R}(U) = S_U^{-1}R$. This is equal to $\mathcal{O}(D_f) = f^{-1}R$. We claim that $\mathcal{O}'$ is a presheaf of rings. All we need to do is give some restriction maps. For $V \subseteq U$, we need to define restriction $\mathcal{O}'(U) \to \mathcal{O}'(V)$. Define $\mathcal{O}'(U) = S_U^{-1}R, \mathcal{O}'(V) = S_V^{-1}R$.

Example 12.1. Let $X = \operatorname{Spec} k[x,y,z,w]$, and $U = X \setminus \{0\}$. Then $\mathcal{O}(\mathbb{A}^2 \setminus \{0\}) = \mathcal{O}(\mathbb{A}^2) = k[x,y]$. (something about codimension one and two), this an example of a scheme that is not affine. We'll return to this later.

Now that we have a presheaf, all that remains is to sheafify. Define $\mathcal{O}_{\operatorname{Spec} R} = (\mathcal{O}'_{\operatorname{Spec} R})^{\text{sh}}$– this is a very nice definition that doesn't show up in textbooks. The reason being this doesn't give the whole picture, since we don't know what its values on the whole space are (global sections change under sheafification).

Definition 12.1. $\mathcal{O}_{\operatorname{Spec} R}$ is a sheaf.

This is a triviality if we use our previous definition.

- $\mathcal{O}(D_f)$ (?)
- these form a basis
- $U \subseteq \operatorname{Spec} R$ open and $U = \bigcup_{i \in I} D_{f_i}$.

This implies we must have

$$\mathcal{O}(U) \to \prod \mathcal{O}(D_{f_i}) \rightrightarrows \prod \mathcal{O}(D_{f_i f_j}).$$

Claim. *This reduces to checming for U distinguished opens and $D = U$.*

We prove this for $U = \operatorname{Spec} R$ itself. We need to show that

$$R \to \prod R_{f_i} \rightrightarrows \prod R_{f_i f_j}$$

where the D_{f_i} cover $\operatorname{Spec} R$, $1 = \sum a_i f_i$. For the identity, to show injectivity suppose $r \in R$ which maps to $\mathcal{O} \in \prod R_{f_i}$. This is equivalent to $r \mapsto 0$ in R_{f_i} for every $i \in I$. Then $R \xrightarrow{f_i} R \xrightarrow{f_i} R \xrightarrow{f_i} \cdots$, and $\frac{r}{1} \in R_{f_i}$ iff $f_i^{N_i} r = 0$ for $N \gg 0$. So $D_{f_i}^{N_i} = D_{f_i}$, and $\bigcup D_{f_i}^{N_i} = \operatorname{Spec} R$ still covers. This implies $1 = \sum b_i f_i^{N_i}$, and $r = r \cdot 1 = \sum b_i f_i^{N_i} r = 0$.

Now we need to show gluing. Given $s_i \in R_{f_i}$ agreeing in $R_{f_i f_j}$, we need to construct $s \in R$ localizing to each s_i. Making a simplifying assumption, suppose $I = \{1, \cdots, n\}$ a finite collection. We have $s_i = \frac{a_i \in R}{f_i^{\ell_i}}, D_{f_i^{\ell_i}} = D_{f_i}$. WLOG, assume $s_i = \frac{a_i}{f_i}$. The s_i, s_j agree in $R_{f_i f_j}$, and since $\frac{a}{b} = \frac{c}{d}$ in R_y, $\frac{a_i}{f_i} \equiv \frac{a_j}{f_j}$ in $R_{f_i f_j}$. Now $g^N(ad - bc) = 0$, so $(f_i f_j)^{m_{ij}}(a_i f_j - a_j f_i) = 0$, $(f_i f_j)^N (a_i f_j - a_j f_i) = 0$. Then $s_i = \frac{a_i}{f_i} = \frac{a_i f_i^N}{f_i^{N+1}} = \frac{b_i}{g_i}$, and $s_i = \frac{b_i}{g_i}$, $D_{g_i} = D_{f_i}$. But now $\frac{b_i}{g_i} = \frac{b_j}{g_j}$ in $R_{g_i g_j}$, $b_i g_j - b_j g_j = 0$, and $\bigcup D_{g_i} = \operatorname{Spec} R$ implies $1 = \sum c_i g_i$. Now we use a trick familiar from topology (partitions of unity); let $s = \sum c_i b_i$. Why is $s|_{D_{g_i}} = s_i$ true? We get

$$\begin{aligned} s \cdot g_j &= \sum c_i b_i g_j \\ &= \sum c_i g_i b_j \\ &= 1 \cdot b_j = b_j, \end{aligned}$$

i.e., $s = \frac{b_j}{g_i}$. If I is infinite, wrote $1 = \sum a_i f_i = \sum_{i \in J \subseteq I \text{ finite}} a_i f_i$. Now we have a finite situation, with $\operatorname{Spec} R = \bigcup_{j \in J} D_{f_j}$. The $(s_i)_{i \in I}$ agree on overlaps, so $(s_i)_{i \in J}$ glues to $s' \in R$. We need s in R_{f_i} to be s_i for all $i \in I$. Fix $\alpha \in I$, and let $J' = J \cup \{\alpha\}$, use J' to construct the gluing. todo:missed this part

13 October 19, 2021

Today we hopefully walk about schemes. What is a ringed space? It is a category with objects being pairs $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of commutative rings on X. For example, we can think about the pairs $(M, \mathcal{O}_M)$ where M is a smooth (resp topological) manifold and $\mathcal{O}_M = C^\infty$ (resp continuous) functions. We are missing the data of what morphisms are. It is obvious to consider continuous maps $f : X \to Y$ between topological spaces, but we need to do something about the functions, and we need a notion of pullback of functions.

If we have $U \subseteq Y$ open for $X \xrightarrow{f} Y$, $f^{-1}(U) \subseteq X$ open, we should have a collection

$$\begin{tikzcd} \mathcal{O}_Y(U) \arrow[r] \arrow[d, "\mathrm{res}"] & \mathcal{O}_X(f^{-1}(U)) \arrow[d, "\mathrm{res}"] \\ \mathcal{O}_V(U) \arrow[r] & \mathcal{O}_X(f^{-1}(V)) \end{tikzcd}$$

We could name this as an f-map from $\mathcal{O}_Y$ to $\mathcal{O}_X$. We can take $\mathcal{O}_X$ and **pushforward** to Y $(\mathcal{O}_Y \to f_x \mathcal{O}_X)$ *or* take $\mathcal{O}_Y$ and **pullback** to X $(f^{-1}\mathcal{O}_Y \to \mathcal{O}_X)$. These notions haven't been defined, but they'll be defined to be compatible with our earlier notion. These two functors are going to be adjoint functors on sheaves of the two spaces. The pushforward is easier to define, but once you know one the other is completely determined by the adjoint.

The definition of pushforward is built to make this work. Given $f : X \to Y$, $f_\cdot : \mathcal{C}_X \to \mathcal{C}_Y$, $F \mapsto f_x F$, define $f_* F(U) := F(f^{-1}(U))$. So a map of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is the data $f : X \to Y$ and $f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X$ a map of sheaves of rings on Y.

Example 13.1. Some examples of pushforwards:

- Suppose we have a map $f : X \to \mathrm{pt}$. We need $f_* F(\mathrm{pt}) = F(f^{-1}(\mathrm{pt})) = F(X) = \Gamma(F)$, the global sections of the sheaf. So global sections are just special cases of the pushforward.
- Suppose we have an inclusion $i : \mathrm{pt} \to X$, so we should get a functor $i_* : \mathcal{C} \to \mathcal{C}_X$. We have

$$i_* F(U) = F(i^{-1}U) = \begin{cases} F(\mathrm{pt}) & \text{if } x \in U, \\ F(\emptyset) & \text{if } x \notin U. \end{cases}$$

  We have alread defined this as $i_* F = \delta_{x,F}$ the skyscraper sheaf of x with value F.

Now let's think about pulling back the same examples. Given $\Gamma = f_*, f : X \to \mathrm{pt}$, we need to find the adjoint.

Lemma 13.1. $\operatorname{Hom}_{\mathcal{C}_X}(\underline{A}, F) = \operatorname{Hom}_{\mathcal{C}}(A, \Gamma(F))$ *where $\underline{A}$ is the constant sheaf with value A. In other words, the operation that takes A to the constant sheaf is the left adjoint to global sections.*

Then the pullback operation is $f^{-1}(A) = \underline{A}$, so the construction of constant sheaves is our first example of pullbacks. For

$$\begin{tikzcd} \underline{A}^{\text{pre}} \arrow[d] \arrow[dr] & \\ \underline{A} \arrow[r] & F \end{tikzcd}$$

we have $\operatorname{Hom}(\underline{A}^{\text{pre}}, F) = \operatorname{Hom}\underline{A}, F)$. So

$$\begin{tikzcd} A = \underline{A}^{\text{pre}}(X) \arrow[r] \arrow[d, hook] \arrow[d, "\text{id}"] & F(X) = \Gamma(X) \arrow[d, "\text{res}"] \\ A = \underline{A}^{\text{pre}}(X) \arrow[r] & F(U) \end{tikzcd}$$

Then for $f : X \to \text{pt}$, $f^{-1} : \mathcal{C} \to \mathcal{C}_X$ is $A \mapsto \underline{A}$, $A^{\text{pre}}(U) = A(f(U))$.

Lemma 13.2. *We have*

$$\operatorname{Hom}_{\mathcal{C}_X}(F, i_*A) = \operatorname{Hom}_{\mathcal{C}}(F_X, A).$$

On $x \in U$, $F(U) \to A$ compatible under restriction is equivalent to $\operatorname{Hom}(\varinjlim_{x\in U} F(U), A)$.

So i^{-1} takes $F \mapsto F_X$ a stalk, and $i^{-1}F(U) \stackrel{?}{=} F(i(U))$ (open?). Pull backs do a thing with shrinking opens and sheafifying, but they're nice as adjoints. BUT on stalks, f^{-1} is the "pullback" of a function: $(f^*g)(x) = g(f(x))$. That is, sheaves are equivalent to functors $X \to \mathcal{C}$, $(f^{-1}F) + x \simeq F_{f(x)}$. Now we go back to rings and stuff.

We are moving toward the "fundamental theorem of algebraic geometry", a correspondence between rings and spaces. Given R a ring, we attached to it a ringed space $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$. For functoriality, we need to talk about maps. Given $\varphi : R \to S$, we have $f = \operatorname{Spec} \varphi : X = \operatorname{Spec} S \to Y = \operatorname{Spec} R$ and $R \to S$ a pullback of functions. We need $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$, i.e., given $r \in R$ we need $D_r \subseteq Y = \operatorname{Spec} R$, where $\mathcal{O}_Y(D_r) \stackrel{?}{\to} \mathcal{O}_X(f^{-1}D_r)$, comparing these two rings gives our pullback. Then $\mathcal{O}_Y(D_r) = r^{-1}R, f^{-1}D_r = D_{\varphi(r)}$, and $\mathcal{O}_X(f^{-1}D_r) = (\varphi(r))^{-1}S$.

We have completed the definition of a functor $\mathsf{Rings}^{\text{op}} \to \mathsf{RingedSpaces}, R \mapsto (\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$. However, we're still missing one last thing. Let $X, \mathcal{O}$ be a "normal" sheaf of functions, $x \in X$, $f \in \mathcal{O}_X$ a stalk of $\mathcal{O}_X$ at x. Suppose $f(x) \neq 0$, then f is *invertible* on U small enough. So $f \in \mathcal{O}_X$ is invertible, and f is not contained in any ideal, i.e., $\mathcal{O}_x$ is a **local ring**. This means there exists a unique maximal ideal ($\mathfrak{m} \subseteq \mathcal{O}_x$ being functions vanishing at x) iff there exists an ideal $\mathfrak{m}$ such that $f \notin \mathfrak{m} \implies f$ is invertible.

Definition 13.1. A **ringed space** $(X, \mathcal{O}_X)$ is a *locally ringed space* (LRS) such that the stalks $\mathcal{O}_X, x \in X$ are local rings.

This implies that for every $x \in X, \mathfrak{m}_x \subseteq \mathcal{O}_x$ is maximal, which implies that $f \in \mathcal{O}(U)$ for $x \in U$. "Evaluate" $f(x) = f \pmod{\mathfrak{m}}$ in $k_x = \mathcal{O}_x/\mathfrak{m}_x$. We say a map of ringed spaces $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is **local** (i.e., a map of LRS) if $f^\#_{y_1} : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is *local*, i.e., maps $\mathfrak{m}_y \to \mathfrak{m}_x$. In other words, we have a map $\mathsf{Rings}^{\text{op}} \to \mathsf{LRS}$.

Observe that $X = \operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R}$ is a LRS with stalks $\mathcal{O} = R_p := (R \setminus p \ni f)^{-1}R, x \in X = \operatorname{Spec} R \iff p \subseteq R$, where $\mathfrak{m}$ is the image of p. Then $\varphi : R \to S$ implies the map $\operatorname{Spec} S \to \operatorname{Spec} R$ is local.

Definition 13.2. A **scheme** is a LRS, locally isomorphic to one of the form $\operatorname{Spec} R$.

So $(X, \mathcal{O}_X)$ has open cover $X = \bigcup U_i$, where each $U_i \simeq \operatorname{Spec} R_i$, $\mathcal{O}_X|_{U_i} \simeq \mathcal{O}_{\operatorname{Spec} R_i}$.

Theorem 13.1. *We have* $\mathsf{Rings}^{\text{op}} \subseteq \mathsf{LRS}$ *a full subcategory of affine schemes. Furthermore, the functor* $\operatorname{Spec} : \mathsf{Rings}^{\text{op}} \to \mathsf{LRS}$ *is the right adjoint to* $\mathsf{LRS} \to \mathsf{Rings}^{\text{op}}$*,* $(X, \mathcal{O}_X) \mapsto \Gamma(\mathcal{O}_X)$*, i.e.,* $(X, \mathcal{O}_X) \to (\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R}) \iff R \to \Gamma(\mathcal{O}_X)$.

This means rings are completely determined by geometric objects, achieving our goal of a faithful dictionary between rings and geometry. Now we can actually start doing algebraic geometry. We'l prove this next time.

14 October 21, 2021

Last time: we had a map R (ring) $\mapsto \operatorname{Spec} R$ (topological space and sheaf $\mathcal{O}_{\operatorname{Spec} R}$), a LRS. This gives rise to a functor $\operatorname{Spec}\colon \mathsf{Rings}^{\mathrm{op}} \to \mathsf{LRS}$.

Definition 14.1. An **affine scheme** is a LRS isomorphic to $\operatorname{Spec} R$ for some R. A **scheme** is a LRS locally of this form, i.e. has an open cover by affines.

Definition 14.2. A **quasi-affine scheme** is a scheme which is isomorphic to an oepn in an affine.

Open subsets of affine are always covered by distinguished opens (spec of localization), so any open subset of an affine is always a scheme, but may not be affine.

Example 14.1. Some examples of schemes that are not affine:

(1) $\mathbb{A}^2_k \setminus \{0\}$ (make it concrete, maybe something like $\mathbb{C}$). Here we consider $\operatorname{Spec} k[x,y] \ni 0 \iff \mathfrak{m}_0 = (x,y)$. The key point is that this ideal is not principal, you need two generators. The claim is that this is a scheme, but not an affine scheme– we can cover $\mathbb{A}^2_k$ by two opens $\mathbb{A}^2 \setminus \{y\text{-axis}\}, \mathbb{A}^2 \setminus \{x\text{-axis}\}$. More concretely, $U_1 = \operatorname{Spec} k[x,y][y^{-1}], U_2 = \operatorname{Spec} k[x,y][x^{-1}]$. Contained in both of these is $U_{12} = \operatorname{Spec} k[x,y][x^{-1},y^{-1}]$. Then
$$\mathcal{O}(\mathbb{A}^2) \simeq \mathcal{O}(\mathbb{A}^2 \setminus 0) = \{f \in \mathcal{O}(U_1), g \in \mathcal{O}(U_2) \text{ agree on overlaps}\} = k[x,y].$$
(Points of $\mathbb{A}^1$ are complex conjugates?)

(2) Consider the projective line $\mathbb{P}^1_k$. How do we define this? Consider $\mathbb{A}^1_x = \operatorname{Spec} k[x], \mathbb{A}^1_y = \operatorname{Spec} k[y]$. There are inclusions $\operatorname{Spec} k[x][x^{-1}], \operatorname{Spec} k[y][y^{-1}]$ (remove the origin), then send identify these two by $x \mapsto y^{-1}$. Glue these two together to get $\mathbb{A}^1_x \amalg \mathbb{A}^1_y$, then $S^2 = \mathbb{C} \cup \infty$ (?). Something about stereographic projection. This is one of the fundamenetal non-affine examples– affine things are great, but projective things are better. We will build more schemes as we go on.

Theorem 14.1. *For X a LRS (eg a scheme), we have*
$$\operatorname{Map}(X, \operatorname{Spec} R) \xrightarrow[\mathcal{O}()]{\simeq} \operatorname{Hom}(R, \mathcal{O}(X))$$
an isomorphism, i.e., $\operatorname{Spec}\colon \mathsf{Rings}^{\mathrm{op}} \to \mathsf{LRS}$ *is the right adjoint to* $\mathcal{O}()\colon \mathsf{LRS} \to \mathsf{Rings}^{\mathrm{op}}$.[4]

Corollary 14.1. $\mathsf{Rings}^{\mathrm{op}} \hookrightarrow \mathsf{LRS}$ *is fully faithful, i.e., maps* $\operatorname{Spec} S \to \operatorname{Spec} R \iff R \to S$*, i.e.,* $\mathsf{AffScheme} \leftrightarrow \mathsf{Rings}^{\mathrm{op}}$.

Given $\varphi\colon R \to \Gamma(\mathcal{O}_X) = \mathcal{O}(X)$, we need to build $X \to \operatorname{Spec} R, x \in X \mapsto ??$ (prime in R), $\varphi^{-1}(\mathfrak{m}_x) \subseteq R$ (functors from R vanish at x). Let $f \in R, \varphi(f) \in \Gamma(\mathcal{O}_X)$. Define $D_{\varphi(f)} = \{x \in X \mid \varphi(f) \text{ doesn't vanish}\}$, then $(\varphi(f))_x \notin \mathfrak{m}_x$. We need to check that $D_{\varphi(f)} = \pi^{-1} D_f$.

Lemma 14.1. *Let X be a LRS, and $s \in \Gamma(\mathcal{O}_X)$. Then $D_S = \{x \in X \mid s(x) \neq 0\}$ is open.*

Proof. Consider $x \in D_s$. Then the germ of s, $(s)_x \in \mathcal{O}_{X,x}$ is invertible (the magic of local rings, either things vanish or are invertible). This means there exists some germ $t \in \mathcal{O}_{X,x}$ where $t \cdot (s)_x = 1$. So there exists some open $x \in U$, where $g \cdot s = 1$, i.e. s is invertible on U. This implies that $x \in U \subseteq D_s$, and we are done. ☒

Corollary 14.2. *s is invertible on all of D_s.*

Proof. Glue inverses. ☒

[4] I missed the entire proof of this, I'll look at Vakil later...

15 October 26, 2021

We have $\mathbb{P}^n$ "defined" as lines through the origin in $\mathbb{A}^{n+1}$. In $\mathbb{P}^1$, say we have $(v,w) \in \mathbb{A}^2$. If $v \neq 0$, then (v,w) is on the same line as $(1, \frac{w}{v})$. Then $\mathbb{A}^1 = \mathbb{P}^1 \backslash \{y \text{ axis}\}$. So $\mathbb{P}^1 = \mathbb{A}^1_{x=1} \cup \mathbb{A}^1_{y=1} = \operatorname{Spec} k[x,y]/(x-1) \cup \operatorname{Spec} k[x,y]/(y-1) = \operatorname{Spec} k[y] \cup \operatorname{Spec} k[x]$. Then we have $\mathbb{P}^1 \setminus \{y \text{ axis}\} \simeq \mathbb{P}^1 \setminus \{x \text{ axis}\} \subseteq \mathbb{A}^1 \setminus 0$, leading to the gluing $\operatorname{Spec} k[x][x^{-1}] \longleftrightarrow \operatorname{Spec} k[y][y^{-1}]$, where $x \mapsto y^{-1}$ and $x^{-1} \mapsto y$.

Consider $A^1 \cup \mathbb{A}^1$, where we glue $\mathbb{A}^1 \setminus 0 \simeq \mathbb{A}^1 \setminus 0$ (gluing everywhere besides the origin), leading to a "UFO" or "ravioli" of sorts. This is a scheme, since it's covered by affine schemes. We have $\mathcal{O}(\text{ravioli}) = k[x] = f \in k[x], g \in k[y]$ which agree under $k[x][x^{-1}]$. This is another example of a scheme, which is not terribly useful besides counterexamples.

Now let's talk about $\mathbb{P}^2$. These are lines in $\mathbb{A}^3$, isomorphic to $\mathbb{A}^2 \cup \mathbb{P}^1$. todo:missed this section

Back to schemes. We have some consequences of this fundamental theorem which says $\operatorname{Map}(X, \operatorname{Spec} R) \simeq \operatorname{Hom}(R, \mathcal{O}(X))$, where X is a scheme. Recall that $\operatorname{Spec} \mathbb{Z}$ is a **final object**, that is, there exists a unique $\operatorname{Map}(X, \operatorname{Spec} \mathbb{Z}) = \operatorname{Hom}(\mathbb{Z}, \mathcal{O}(X)), 1 \mapsto 1$. In the case of $\operatorname{Spec} \mathbb{Z}$,for a map $\mathbb{Z} \to \mathcal{O}(X)$, we can ask whether the map factors through $\mathbb{Z}/p$ or not.

$$\begin{tikzcd} \mathbb{Z} \arrow[rr] \arrow[dr] & & \mathcal{O}(X) \\ & \mathbb{Z}/p \arrow[ur, dotted] & \end{tikzcd}$$

If it does, we say X has **characteristic** p. Now consider $\operatorname{Map}(X, \mathbb{A}^1_{\mathbb{Z}}) = \operatorname{Hom}(\mathbb{Z}[x], \mathcal{O}(X))$. A map from $\mathbb{Z}$ has no data (determined by $\mathbb{Z}$), but for $\mathbb{Z}[x]$ homomorphisms are completely determined by the image of x, so this is precisely $\mathcal{O}(X) \simeq \mathbb{A}^1_2 \times (A^1_y \setminus 0)$. The point is that maps to $\mathbb{A}^1$ *are* functions. Why are the $\mathcal{O}(X)$ a ring? Rather, why are maps $\operatorname{Map}(X, \mathbb{A}^1)$ a ring? The cool answer is that $\mathbb{A}^1$ itself has extra structure, we can add and multiply in $\mathbb{A}^1$ (we say $\mathbb{A}^1$ is a "ringed scheme"). For example, we have an interesting map $\mathbb{R} \times \mathbb{R} \xrightarrow{(x,y) \mapsto x+y} \mathbb{R}$, $(+)^* f(x,y) = f(+(x,y)) = f(x+y)$. Then $\mathbb{A}^2 \simeq \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$, iff $\mathcal{O}(\mathbb{A}^1) \to \mathcal{O}(\mathbb{A}^1 \times \mathbb{A}^1)$, where $k[x] \mapsto k[u,v]$, and $x \mapsto u + v$. There is an analogous idea with multiplication $x \mapsto u \cdot v$ and the inverse map $(-\colon \mathbb{A}^1 \xrightarrow{x \mapsto -x} \mathbb{A}^1)$. Paired with $\mathcal{O}\colon \operatorname{Spec} \mathbb{Z} \to \mathbb{A}^1$, these satisfy the axioms of a grouop in the category Sch, and form a **group object**.

What is a group object? These make sense in any category– for $G \in \mathcal{C}$, we need maps $G \times G \xrightarrow{+} G, G \xrightarrow{-} G, x \xhookrightarrow{0} G$. For example, in Set we have ordinary groups, in Top we have topological groups, and in the category of smooth manifolds we have Lie groups. For $X \in \mathcal{C}$, $\operatorname{Hom}(X,G) \in \mathsf{Set}$ means $\operatorname{Hom}(X, G \times G) \to \operatorname{Hom}(X,G) = \operatorname{Hom}(X,G) \times \operatorname{Hom}(X,G)$.

Proposition 15.1. *Let $G \in \mathcal{C}$ be an object in a category $\mathcal{C}$. Then TFAE:*

(1) G is a group object,

(2)

$$\begin{tikzcd} \underset{x \mapsto \operatorname{Hom}(X,G)}{h_G = \operatorname{Hom}(-,G)\colon \mathcal{C}} \arrow[r] \arrow[dr, dotted] & \mathsf{Set} \\ & G_p \arrow[u, "\text{forget}"'] \end{tikzcd}$$

In other words, this makes $\operatorname{Map}(X,G)$ into a group functorially.

Here for $X \to Y$, $h_G(Y) \to h_G(X)$, and $h_G(x)$ is a group. Then $\mathbb{A}^1, +$ is a group scheme iff $\operatorname{Map}(X, \mathbb{A}^1, +) = \mathcal{O}(X), +$.[5] We claim that $\mathbb{A}^1 \backslash 0$ is a group scheme (recall $\mathbb{A}^1$ is a monoid with multiplication $k[x] \to k[u,v], x \mapsto u \cdot v$).

[5] Ring schemes are never used but group schemes come up a lot.

We have $\mathbb{A}^1 \setminus 0 = \operatorname{Spec} k[x][x^{-1}] \xrightarrow{(-)^{-1}} \mathbb{A}^1 \setminus 0$ with inverse $x \mapsto x^{-1}$. This group scheme is so useful we denote it by $\mathbb{G}_m = \mathbb{A}^1 \setminus 0$, $\mathbb{G}_a = \mathbb{A}^1, +$. Then $\operatorname{Map}(X, \mathbb{G}_m), x \mapsto f = \operatorname{Hom}(k[x, x^{-1}], \mathcal{O}(X)) = \{f \in \mathcal{O}(X) \mid f^{-1} \text{ exists}\} = \mathcal{O}^\times$ units. So a formula for $\mathbb{P}^n$ living in $\mathbb{A}^{n+1} \setminus 0$ is $\{\mathbb{A}^{n+1} \setminus 0\}/\mathbb{G}_m$. Note that $\operatorname{Map}(\operatorname{Spec}\mathbb{Z}, \mathbb{G}_m) = \mathbb{Z}^\times = \{\pm 1\}$.

16 October 28, 2021

Following an explanation:

> "I'm more confused." –Student
> "You're more confused. Great." –Ben-Zvi

Today we'll motivate one of the most important constructions in algebraic geometry, fiber products. Recall that a scheme $X = \operatorname{Spec} R$ and a map $f \in \mathcal{O}(X) = R$ is equivalent to a map $f : X \to \mathbb{A}^1 = \operatorname{Spec}\mathbb{Z}[x]$, where $\mathbb{Z}[x] \xrightarrow{x \mapsto f} \mathcal{O}(X)$. There is the vanishing locus $V(f) = f^{-1}(0)$, which is isomorphic to $\operatorname{Spec} R/f$. Here $f^{-1}(0) = \{p \subseteq R \mid f \in p\}$. Something happened.

Theorem 16.1. Sch *has fiber products.*

$$\begin{tikzcd} & X \times_Z Y \arrow[r] \arrow[d, dashed] & X \arrow[d] \\ U \arrow[ur, dotted] \arrow[urr] \arrow[r] & Y \arrow[r] & Z \end{tikzcd} \qquad \begin{tikzcd} \operatorname{Hom}(U, X \times_Z Y) \arrow[r] \arrow[d] & \operatorname{Hom}(U, X) \arrow[d] \\ \operatorname{Hom}(U, Y) \arrow[r] & \operatorname{Hom}(U, Z) \end{tikzcd}$$

Why is this relevant? Fibers are a special case of fiber products. The fundamental theorem says $\mathsf{Aff} = \mathsf{Rings}^{\mathrm{op}}$ is the right adjoint of $X \mapsto \mathcal{O}(X)$, which preserves all limits. This is the same as a limit in $\mathsf{Rings}^{\mathrm{op}}$ which is a colimit in Rings. A digression on tensor products.

We defined how to take the tensor product for modules of a ring, given M, N A-modules for A a ring, we defined $M \otimes_A N$ as a colimit, that is, whenever M, N mapped somewhere both A-linearly, then it factors through $M \otimes_A N$.

$$\begin{tikzcd} N \arrow[dr] \arrow[drr] & & \\ & M \otimes_A N \arrow[r, dotted] & U \\ M \arrow[ur] \arrow[urr] & & \end{tikzcd}$$

Now for $A \to B$ a homomorphism of rings (B is an A-algebra), then $B \otimes_A N$ is a B-module. We have $B \otimes_A (-)$: A-modules $\to$ B-modules (symmetric monoidal functor that takes rings to rings), then $B \otimes_A C$ is a ring with $(b_1 \otimes c_1) \cdot (b_2 \otimes c_2) = (b_1 b_2) \otimes (c_1 c_2)$, i.e., the tensor product gives pushouts (fibered coproducts) in Ring. This is also known as the tensor product over A-algebras.

$$\begin{tikzcd} & B \arrow[dr] \arrow[drr] & & \\ A \arrow[ur] \arrow[dr] & & B \otimes_A C \arrow[r, dotted, "\text{ring}"] & U \ \text{ring} \\ & C \arrow[ur] \arrow[urr] & & \end{tikzcd}$$

This is to say rings have a natural operation of a tensor product, and the fundamental theorem immediately implies that schemes have fibered products and we know what they are. That is,

$$\begin{tikzcd} & \operatorname{Spec} B \times_{\operatorname{Spec} A} \operatorname{Spec} C = \operatorname{Spec} B \otimes_A C. \arrow[dl] \arrow[dr] & \\ \operatorname{Spec} B \arrow[dr] & & \operatorname{Spec} C \arrow[dl] \\ & \operatorname{Spec} A & \end{tikzcd}$$

i.e., $\operatorname{Map}(X, \operatorname{Spec} B \otimes_A C) = \{\text{maps } X \to \operatorname{Spec} B, \operatorname{Spec} C \text{ agreeing on } X \to \operatorname{Spec} A\}$. Then $\operatorname{Spec} R \to \mathbb{A}_k = \operatorname{Spec} k[x] \xleftarrow{0} \operatorname{Spec} k$, $k[x] \to k, x \mapsto 0$. So $\operatorname{Spec} R \otimes_{k[x]} k = R/f \to \operatorname{Spec} R$.

> "The non-existence of a zero object means we're happy, because it means we're doing geometry, not linear things. Linear things are a shadow of non-linear things."

Now $R \otimes_{k[x]} k = \{r \otimes \lambda\} = R = R \otimes k/x \cdot r = fr \otimes \lambda = r \otimes x \cdot \lambda$. Here $x \otimes \lambda = 0$, so $fr \otimes \lambda = 0$, and $R \otimes_{k[x]} k \simeq R/f$. Something happened? $\mathbb{A}^1 \times_{\operatorname{Spec} \mathbb{Z}} \mathbb{A}^1 = \mathbb{A}^1 \times \mathbb{A}^1 = \operatorname{Spec} \mathbb{Z}[x] \otimes_{\mathbb{Z}} \operatorname{Spec} \mathbb{Z}[y] = \operatorname{Spec} \mathbb{Z}[x, y] = \mathbb{A}^2$. We defined a scheme as an abstract object, but secretly we should think of it as a scheme over something. Topological spaces can be thought of as maps to a point, but that gives no information. However, the final object for Sch is $\operatorname{Spec} \mathbb{Z}$, which is an interesting object.

Let's do one more example. Last time we talked about whether or not a scheme has a certain characteristic. Consider $\mathbb{A}^1_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$. Inside of $\operatorname{Spec} \mathbb{Z}$ we have $\mathbb{Z}/5$

17 November 2, 2021

todo:missed this lecture

18 November 4, 2021

todo:missed the first 20 minutes of this class. Recall the base change $\mathrm{Sch}/R \xrightarrow{\times_{\operatorname{Spec} R} \operatorname{Spec} T} \mathrm{Sch}/T$, which is just the ordinary tensor product for rings.

$$\begin{tikzcd} & X \times_{\operatorname{Spec} R} \operatorname{Spec} T \arrow[dl] \arrow[dr] & \\ X \arrow[dr] & & \operatorname{Spec} T \arrow[dl] \\ & \operatorname{Spec} R & \end{tikzcd} \qquad \begin{tikzcd} & S \otimes T & \\ S \arrow[ur] & & T \arrow[ul] \\ & R \arrow[ul] \arrow[ur] & \end{tikzcd}$$

In particular, we can turn any scheme into one of characteristic p, $\mathrm{Sch} \to \mathrm{Sch}/\mathbb{Z}/p$. Here the fiber product is just the fiber. Intersections in algebraic geometry are just tensor products. Now consider $U \subseteq Z$, and a fiber product $Z \leftarrow X \leftarrow X \times_Z U \to U = \operatorname{Spec} \mathbb{Z}$. Then the fiber product with opens inside X.

Why are we doing this? Schemes are always schemes over $\operatorname{Spec} \mathbb{Z}$, or $\mathrm{Sch} = \mathrm{Sch}/\operatorname{Spec} \mathbb{Z}$. Somehow the geometry algebraic geometry is assembling all the X_p's (the fibers) into something. Even if we don't are about arithmetic things ($\operatorname{Spec} \mathbb{Z}$), doing geometry in this settting naturally leads to others. For example, consider $R = \mathbb{C}[x]$. Then for Sch$/R$ a scheme over R, we think about a parameter of schemes (?). For example, consider $y^2 = x^3 + \lambda x\mu$, thought of as a family in $\mathbb{A}^2$ with two parameters. So we think of $\mathbb{C}[\lambda,\mu]$-algebras or Sch$/\mathbb{A}^2_{\mathbb{C}}$.

$$\begin{tikzcd}
\operatorname{Spec}\mathbb{C}[x,y,\lambda,\mu] = \mathbb{A}^2_S \arrow[d] & \supset & \operatorname{Spec}\mathbb{C}[x,y,\lambda\mu]/y^2 = x^3+\lambda x+\mu \arrow[dll] \\
\operatorname{Spec}\mathbb{C}[\lambda,\mu] & &
\end{tikzcd}$$

Back to the idea of group schemes/ A group scheme over R is a group object in Sch$/R$, or some $G \in$ Sch$/R$ with a map $G \times G \to G$, which is a product in Sch$/R$.

Claim. *This product in* Sch$/R$ *is the exactly the fiber product, where* $X, Y \mapsto X \times_{\operatorname{Spec} R} Y$.

The reason is that base changes preserve limits (fiber products). Now let's talk about the identity. We want to say that it's a point of G, what does that mean? We claim that the identity e is a map $e\colon \{\text{final object}\} = \operatorname{Spec} R \to G$. Why is this a thing? One way to think about it is on the level of functors of points. $\mathrm{Map}_{\mathrm{Sch}/R}(X, G)$ should be a group, and $M\mathrm{Map}(X, \text{final object}) = x$. Multiplication is defined on each fiber, and the identity is not a point, but a whole section.

In algebraic geometry land we could say the base is $\mathbb{A}^1$ 0, or in topology land S^1. Consider $\mathbb{Z}/3$ a point, which is the smallest group with an interesting automorphism. This corresponds to (something?) on the circle, corresponding with a three-fold cover.